THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Volume 2 *No.* 5 *March* 1931

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OXFORD

AT THE CLARENDON PRESS

1931

Price 7s. 6d. net

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

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Papers on subjects of Pure and Applied Mathematics are invited and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Mr. Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E.C. 4).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C. 4

THE QUARTERLY JOURNAL
OF
MATHEMATICS

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VOLUME II

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OXFORD UNIVERSITY PRESS
AMEN HOUSE, E.C. 4
LONDON EDINBURGH GLASGOW
LEIPZIG NEW YORK TORONTO
MELBOURNE CAPETOWN BOMBAY
CALCUTTA MADRAS SHANGHAI
HUMPHREY MILFORD
PUBLISHER TO THE
UNIVERSITY

PRINTED IN GREAT BRITAIN AT THE UNIVERSITY PRESS, OXFORD
BY JOHN JOHNSON, PRINTER TO THE UNIVERSITY

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CORRIGENDA

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Page 252: in first line of footnote §, for s read u

Page 253: replace last formula by $F(t) = t^{-1} \int_0^t f(u)\,du$

Page 258: in seventh line from the end, for $\sum_{\nu=0}^{r}$ read $\sum_{\nu=0}^{r}$

Page 262: in the right-hand member of (4.1), for $2(k+1)$ read $xk(k+1)$

Page 266: in penultimate line, for $(1-e^{-u})^{1-\rho}$ read $(1-e^{-u})^{1+\rho}$

Page 271: in line ten, for $\{\sin\frac{1}{2}(t+mx)\}^{3+\rho}$ read $\{\sin\frac{1}{2}(t+mx)\}^{2+\rho}$
in line thirteen, for $\{\sin\frac{1}{2}(t+mx)\}^{2+\rho}$ read $\{\sin\frac{1}{2}(t+mx)\}^{3+\rho}$

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Page 155: in (3), for $\sum_{r=1}^{n}$ read $\sum_{r=1}^{\infty}$

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Page 209: in (1.22), for $\Phi_{\alpha,\beta}$ read $\Phi'_{\alpha,\beta}$

Page 213: in line six, delete *then*

Page 219: in (4.21), for $\Gamma(\alpha)$ read $(\log C)^{\beta}\Gamma(\alpha)$

Page 224: in line fourteen, for (1.2) read (1.14)
in (5.44), for $C_n(t)$ read $c_n(x)$
in (5.45), for C_n read $c_n(x)$
in the last line, for $-\sum_{n=1}^{\infty}$ read $-t\sum_{n=1}^{\infty}$
for $-\int_0^{\infty}$ read $-t\int_0^{\infty}$

Page 227: in the third line of Theorem 6.1 and Theorem 6.2, for $\beta > 0$ read $\beta \geqslant 0$

Page 228: in (6.23), for $\Phi_{\alpha,\beta}$ read $\Phi'_{\alpha,\beta}$

A THEOREM ON PERIODIC TRANSFORMATIONS OF SPACES

By M. H. A. NEWMAN

[Received 1 August 1930]

THE object of this paper is to prove that locally Euclidean spaces admit no infinitesimal transformations with an assigned period p, i.e.

THEOREM 1. *If M_n is a locally Euclidean metricized connected n-dimensional space, Ω_n any domain in it, and p an integer greater than* 1, *there is a positive number d such that no uniform continuous representation of M_n on itself with period p moves every point of Ω_n a distance less than d.*

The *period* of a representation f is the least integer greater than 0 such that $f^p(x) = x$ for all points x of M_n, where

$$f^p(x) = f(f(f...(f(x))...)) \quad (p \text{ of them}).†$$

A periodic uniform transformation is necessarily (1, 1), for if $f(x_1) = f(x_2)$, then $f^p(x_1) = f^p(x_2)$.

In order to make it easier to picture the symmetry properties used in the proof, we shall consider in detail only the case $p = 2$. Formally, this case has no special features, and a few words at the end will make clear the modifications necessary in the general case.

Let $\rho(x, x')$ be the distance between the points x, x' in the given metric on M_n. (This metric, which is one of the data of the problem is to be distinguished from the local 'Euclidean distance' $\sqrt{\sum (x'_i - x_i)^2}$ depending on the choice of a coordinate system.) Let U be a Euclidean neighbourhood contained in Ω_n, (x_i) a set of Euclidean coordinates in U such that all the points for which $|x_i| \leqslant 3$ are present in U.

Let d_1 be the distance between the closed set $M_n - U$ and the set $-2 \leqslant x_i \leqslant 2$; let d_2 be the lower bound of $\rho(x, x')$ for (x) and (x') in U such that $\sum (x_i - x'_i)^2 \geqslant \frac{1}{4}$; and let d be a positive number less than the smaller of d_1 and d_2.

ASSERTION A. *If f, of period* 2, *moves every point of U a distance less than d, all points of the cube $-1 \leqslant x_i \leqslant 1$ are fixed points of f.*

Suppose (x_i^0) of the cube $-1 \leqslant x_i \leqslant 1$ is not a fixed point. If C is the cube $-1 < x_i - x_i^0 < 1$, an open set, then $C \subset \{-2 \leqslant x_i \leqslant 2\}$ and

† It is convenient to interpret $f^0(x)$ as the identical transformation.

so $f(C) \subset U$, since $d < d_1$. The second condition on d then shows that $|f_i(x) - x_i| < \frac{1}{2}$ for all x of C, where $f_i(x)$ is the i-coordinate of $f(x)$, and hence that the degree (*Abbildungsgrad*) of f is 1, and that $C + f(C) \subset \{-3 \leqslant x_i \leqslant 3\}$. Since $f[C+f(C)] = f(C)+C$, the set $C+f(C)$ is represented on itself, and we may confine our attention to it, supposing it embedded in the complete R_n (Euclidean n-space), instead of merely in U. Further, we may suppose x_i^0 to be the origin in R_n.

Consider the $2n$-dimensional product space $R_n \times R_n$, with coordinates (x_i, y_i) $(i = 1, 2, ..., n)$. When x moves in $C+f(C)$, the point $x \times f(x)$ of $R_n \times R_n$, i.e. the point with coordinates $[x_i, f_i(x)]$, traces out an n-dimensional space K_n, homœomorphic to $C+f(C)$.† The periodicity of f finds expression in a symmetry property of K_n. For the necessary and sufficient condition that (x_i, y_i) lie on K_n is that $(x_i) \subset C+f(C)$ in R_n, and that $y_i = f_i(x)$. But in these circumstances $y_i \subset C+f(C)$ and $f_i(y) = f_i^2(x) = x_i$: the point (y_i, x_i) also lies on K_n, so that K_n is symmetrical about the flat n-fold $x_i = y_i$.

If $C+f(C)$ is simplicially divided into an enumerable infinity of simplexes so that only a finite number are contained in any closed set in $C+f(C)$, then with any n-simplex $\{x\}$‡ of $C+f(C)$ we may correlate the n-simplex $\{x \times f(x)\}$ of K_n, which is therefore an open manifold. We project K_n on to the flat n-fold $x_i = y_i$ by the rule

$$\phi: \qquad (x, y) \to \left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

This representation has a finite degree near the origin. For the boundary of K_n (i.e. the set $\bar{K}_n - K_n$ in $R_n \times R_n$) is contained in the set $\{x \times f(x)\} + \{f(x) \times x\}$, where x moves on the boundary of the cube C. All such points x have at least one coordinate $= \pm 1$, say $x_j = 1$. Since $|x_j - f_j(x)| < 1$, $f_j(x) > 0$ and therefore $\frac{1}{2}(x_j + f_j) > \frac{1}{2}$; no boundary point projects into the part of $x_i = y_i$ in which $|x_i| \leqslant \frac{1}{2}$.

We estimate the degree of ϕ at the origin in two ways:

(i) The part of $x_i = y_i$ within $C \times C$, i.e. within $|x_i| \leqslant 1$, $|y_i| \leqslant 1$, is mapped uniformly on $x_i = y_i$ by the rule

$$\psi: \qquad (x, x) \to \left(\frac{x+f(x)}{2}, \frac{x+f(x)}{2}\right).$$

† This is the manifold introduced by Lefschetz as a 'graph' of the representation f.

‡ $\{x\}$ means 'a set of points of which x is a typical member'.

In this mapping, for every i

$$|\psi_i(x)-x_i| = \tfrac{1}{2}|f_i(x)-x_i| < \tfrac{1}{2},$$

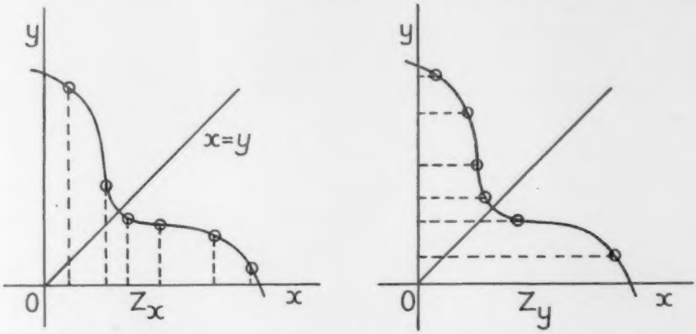
and therefore by a theorem of Brouwer, the degree of ψ at 0 is 1. But ψ may be carried out in two steps:

$$(1)\ (x,x) \to (x,f(x)),$$

$$(2)\ (x,f(x)) \to \left(\frac{x+f(x)}{2}, \frac{x+f(x)}{2}\right).$$

Step (1) is a (1, 1) representation of $x_i = y_i$ on K_n, and has therefore the degree ± 1; step (2) is ϕ. Hence ϕ has the degree ± 1.

(ii) We shall now show, by using the periodicity of f, that the degree of $\phi \equiv 0 \pmod 2$, a contradiction which shows the falsity of



our original assumption that (x^0), i.e. the origin in our new coordinates, is not a fixed point. From the intuitive conception of the degree as the number of times K_n covers $x_i = y_i$, it appears obvious that the projection of any manifold K_n on a flat variety about which it is symmetrical must have an even degree, for the points of K_n can be grouped into pairs which fall on the same point of the flat variety. But the formal proof, using Brouwer's approximate definition, is somewhat difficult, since we have no method of simplicially dividing K_n so that the simplexes themselves are symmetrically disposed. It is therefore necessary† to make a preliminary deformation of K_n.

According to our assumption, O, the origin in $R_n \times R_n$, is at a positive distance a from K_n. Let L_n be the part of K_n satisfying $|x_i+y_i| < b$, where $b < \frac{1}{2}$ and also b is so small that if $(x,y) \subset L_n$,

† The use of Hopf's definition of the degree, *Math. Ann.* 100 (1929), p. 579, does not, I think, avoid this difficulty.

then $|x_i - y_i| \geqslant \frac{1}{2}a$. Let X_{2n} be the part of $R_n \times R_n$ whose points satisfy $|x_i + y_i| < b$. Divide $R_n \times R_n$ into 'cubes' by the $(2n-1)$-folds

$$x_i = r\epsilon, \qquad y_j = s\epsilon \qquad (r, s = 0, \pm 1, \ldots)$$

ϵ being so small that, if $(x, y) \subset L_n$, (x, y) and (y, x) do not belong to adjacent cubes.

A simplicial subdivision of K_n can be derived in two ways from that of $C + f(C)$, viz. first by taking as simplexes the sets $\{x \times f(x)\}$, where $\{x\}$ is a simplex of $C + f(C)$, secondly by taking the sets $\{f(x) \times x\}$. We shall call these two simplicial divisions Z_x and Z_y (see fig. 1).

Let C_{2n}^1 be an ϵ-cube containing a point of L_n within it, and let (a_i, b_i) be the centre of C_{2n}^1. Since K_n is a non-singular n-manifold in $R_n \times R_n$, there is a point (a_i', b_i') arbitrarily near (a_i, b_i) not on K_n. From this point, chosen within C_{2n}^1, project the part of K_n in C_{2n}^1 on to the boundary of C_{2n}^1. From the symmetry of K_n the cube C_{2n}^2 with centre (b_i, a_i) contains a point of L_n, and (b_i', a_i') is not on K_n. Project from (b_i', a_i') on to the boundary of C_{2n}^2. Proceeding in this way through all the cubes containing a point of L_n, we obtain a continuous image $K_n^{(1)}$ of K_n under a certain mapping function θ_1. $K_n^{(1)}$ is, like K_n, symmetrical about $x_i = y_i$, and all its points in X_{2n} are on $(2n-1)$-faces of ϵ-cubes.

The next step is to project $K_n^{(1)}$ on to $(2n-2)$-faces of ϵ-cubes. This needs a little preparation since $K_n^{(1)}$ may cover every point of certain $(2n-1)$-cubes. Let C_{2n-1}^1 be any $(2n-1)$-cube containing a point of $K_n^{(1)}$, (a_i'', b_i'') its centre. Choose one of the two simplicial subdivisions of K_n, say Z_x, and call 'σ-simplexes' those n-simplexes of $K_n^{(1)}$ containing either (a_i'', b_i'') itself or all the vertices of a flat n-simplex containing (a_i'', b_i''). Suppose the original simplicial division $C + f(C)$ was so fine that all σ-simplexes, and all n-simplexes of $K_n^{(1)}$ adjacent to them, lie in C_{2n-1}^1. Replace every σ-simplex by the rectilinear simplex with the same vertices; and more generally, every simplex of which the k vertices $p_0, p_1, \ldots, p_{k-1}$ belong to σ-simplexes (while $p_k, \ldots, p_n$ do not) by the join of the flat $(k-1)$-simplex $p_0 p_1 \ldots p_{k-1}$ to the original residual $(n-k)$-simplex $p_k \ldots p_n$—i.e. by the set of points generated by the flat simplex $\xi p_0 \ldots p_{k-1}$ when ξ moves over the 'curved' simplex $p_k \ldots p_n$. Since all the simplexes of $K_n^{(1)}$ that are affected lie in C_{2n-1}^1, so do the modified simplexes, and by the definition of σ-simplexes, a_i'', b_i'' is covered, after the modifica-

tion, only by rectilinear simplexes (if at all). Hence there is a point (a_i''', b_i''') of C^1_{2n-1} arbitrarily near (a_i'', b_i'') not covered at all by the modified $K_n^{(1)}$. From (a_i''', b_i''') we project the part of the modified manifold in C^1_{2n-1} on to the boundary of C^1_{2n-1}.

The part of $K_n^{(1)}$ in C^2_{2n-1}, the cube with centre (b_i'', a_i''), is now treated exactly symmetrically, the simplicial division Z_y being used for the preliminary deformation. This will not affect what has been done in C^1_{2n-1} since the two $(2n-1)$-cubes have no common point. All the other $(2n-1)$-faces of ϵ-cubes are to be treated, in pairs, in the same way, and the resulting continuous image of K_n is to be called $K_n^{(2)}$, and the mapping function θ_2.

Thus, step by step, we project $K_n^{(1)}$ on to the $(2n-2)$-faces, the result, $K_n^{(2)}$, on to the $(2n-3)$-faces, and finally $K_n^{(n-1)}$ on to the n-faces of the ϵ-cubes. The result is a continuous image, $K_n^{(n)}$, of K_n, under a mapping function θ_n which moves each point of K_n a distance $\epsilon\sqrt{2n}$ or less. $K_n^{(n)}$ is symmetrical about $x_i = y_i$, and the part of it in X_{2n} lies on the n-faces of ϵ-cubes.

Since the degree of f is 1, the indicatrices $[p_0 p_1 \dots p_n]$ and $[f(p_0) f(p_1) \dots f(p_n)]$ for a simplex of C and its image under f are coherent. Hence symmetrical indicatrices for symmetrically situated simplexes of the manifold K_n are coherent within the orientable sub-space L_n of K_n. If, then, symmetrical indicatrices are assigned to symmetrical pairs of n-faces of ϵ-cubes, the degree of θ_n is the same, including the sign, in the two members of a pair.†

Project $K_n^{(n)}$ on to $x_i = y_i$ by the rule

$$\phi: \qquad (x, y) \to \left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

This gives a representation $\phi\theta_n$ of the original K_n on the same n-fold. Let x^* be a point of $x_i = y_i$, very near O but such that no point of any $(n-1)$-face of an ϵ-cube is projected on to it. Owing to the indexing rule we have adopted, symmetrical pairs of n-faces that cover x^* cover it with the same degree, and they are covered by K_n with the same degree c_r. Hence the degree of $\phi\theta_n$ at O is equal to $\sum \pm 2c_r$, i.e. is even. Since θ_n moves no point of K_n more than $\epsilon\sqrt{2n}$, ϕ and $\phi\theta_n$ have the same degree when ϵ is small enough. Hence degree $(\phi) \equiv 0 \pmod 2$ at O.

The contradiction at which we have now arrived establishes Asser-

† This attention to the sign, though superfluous when $p = 2$, is necessary for larger values of p.

tion A, that the points $-1 \leqslant x \leqslant 1$ are fixed points for all transformations of period p of M_n which move every point of U a distance less than d. To complete Theorem 1 for the case $p = 2$ we must show that such transformations leave all points of M_n fixed. For then, given Ω_n, we may take $d(\Omega_n)$ to be $d(U)$, determined as on p. 1 for any U contained in Ω_n.

The required result is a consequence of the following theorem.

THEOREM 2. *If a uniform continuous transformation of M_n of period 2 leaves the points of a domain D fixed, it leaves all points of M_n fixed.*

Let U be any connected Euclidean neighbourhood overlapping D. To establish Theorem 2, it is sufficient to show that from the fact that D is fixed it follows that all points of U are fixed. For if, then, U_1 is any neighbourhood overlapping U, it will follow similarly that all points of U_1 are fixed, and hence, step by step, all points of M_n.

It is sufficient, then, to show that, if (x_i) is a coordinate system in U, and C is the set of points $0 \leqslant x_i \leqslant \epsilon$ contained in D, then all the points of those *complete* cubes in U, bounded by

$$x_i = r\epsilon \qquad (r = 0, \pm 1, \ldots)$$

that are $(n-1)$-dimensionally connected with C, are themselves fixed points, however small ϵ may be.

Suppose that for some ϵ this is not so. Then there is a cube, which by a change of coordinates we may take to be

$$0 \leqslant x_i \leqslant \epsilon,$$

whose points are all fixed, while it has a neighbour, say

$$\epsilon \leqslant x_1 \leqslant 2\epsilon, \qquad 0 \leqslant x_j \leqslant \epsilon \qquad (j = 2, \ldots, n),$$

of which some points move.

Let η be the upper bound of numbers ξ such that the points of the set

$$\Delta_\xi : \qquad \left.\begin{array}{l} 0 \leqslant x_1 \leqslant \xi \\ 0 \leqslant x_j \leqslant \epsilon \end{array}\right\}$$

are all fixed. By hypothesis η exists and $\epsilon \leqslant \eta < 2\epsilon$. At least one point of $x_1 = \eta$ is a limit point of points moved by f. Let (x_i^0) be such a point, $(x_1^0 = \eta)$, and first suppose (x_i^0) is interior to the face $x_1 = \eta$ of Δ_η. If θ is a small enough positive number, the cube

$$C_\theta : \qquad -\theta \leqslant x_i - x_i^0 \leqslant \theta$$

lies entirely in the sum of the two ϵ-cubes and is bisected by $x_1 = \eta$. If x is a point on the boundary of C_θ for which $x_1 > \eta$, x is certainly

not transformed into its reflection in the point (x_i^0), for its reflection has $x_1 < \eta$, and so is transformed into itself. Hence, for no point x_i of the boundary of C_θ is

$$\tfrac{1}{2}\{x_i+f_i(x)\} = x_i^0.$$

If, then, we construct K_n in $R_n \times R_n$ as before, the representation of K_n on $x_i = y_i$

$$\phi: \qquad (x_i, y_i) \to \left(\frac{x_i+y_i}{2}, \frac{x_i+y_i}{2}\right)$$

used in Theorem 1, does not project any boundary point of C_θ on to (x_i^0, x_i^0). The degree of ϕ is therefore finite and constant in a small neighbourhood of (x_i^0, x_i^0). But for points $x_i < x_i^0$ the transformation is the identity, and the degree is 1, while if (x_i) is not a fixed point, the argument of Theorem 1 shows that at (x_i, x_i) the degree $\equiv 0 \pmod 2$. Hence the assumption that there are points arbitrarily near (x_i^0) which are moved by f leads to a contradiction.

It was assumed that x_i^0 is interior to the face $x_1 = \eta$ of the rectangular block Δ_η. If it is on the boundary of this face but interior to the $(n-2)$-component $x_1 = \eta$, $x_2 = \epsilon$, we prove by exactly the same argument that all points of

$$\Delta': \qquad \begin{aligned} &|x_2 - x_2^0 + \theta| \leqslant \theta \\ &|x_i - x_i^0| \leqslant \theta \quad (i = 1, 3, 4, \ldots n) \end{aligned}$$

are fixed, for some positive θ. But (x_i^0) is interior to a face of Δ', viz. $x_2 = x_2^0$, and therefore the argument of the previous case may now be used, Δ' taking the place of Δ_η, to show that all points in a neighbourhood of (x_i^0) are fixed. If (x_i^0) belongs to an $(n-k)$-component but not to an $(n-k-1)$-component of Δ, there exists a small cube, interior to U, and half in and half out of Δ, such that (x_i^0) lies in an $(n-k+1)$-component but not an $(n-k)$-component of the cube; and the original argument shows that all points of this cube are fixed points of f. Hence, if we make the inductive hypothesis that the case $n-k+1$ is settled, it follows that in the case $n-k$ also all points near (x_i^0) are fixed points of f.

This completes the proof of Theorem 2, and with it of Theorem 1 for the case $p = 2$.

Of the case $p > 2$ little need be said. We define d_1 and d_2 as before, and d as the smaller of d_1/p and d_2/p. If $\rho(f(x), x) < d$ for all x in U, then (the coordinate system x_i satisfying the same condi-

tions as before) the assumption that when x lies in C, $f^r(x)$ lies in $\{-3 \leqslant x_i \leqslant 3\}, \subset U$, for $r = 1, 2,..., s-1$, gives

$$\rho(f^s, x) \leqslant \sum_0^{s-1} \rho(f^{j+1}, f^j) < sd \leqslant d_1 \text{ and } d_2.$$

Hence $f^s(x)$ also lies in $\{-3 \leqslant x_i \leqslant 3\}$, i.e.

$$\sum_0^{p-1} f^r(C) \subset \{-3 \leqslant x_i \leqslant 3\}.$$

We now consider the product space $R_n \times R_n \times ... \times R_n$ (p of them) with coordinates

$$x_i^j \quad (j = 1, 2,..., p;\ i = 1, 2,..., n).$$

The locus of the point $x \times f(x) \times ... \times f^{p-1}(x)$ when x moves over $\sum_0^{p-1} f^r(C)$ is an n-dimensional manifold K_n, homœomorphic to $\sum f^r(C)$ and symmetrical about the n-fold

$$I_n: \qquad x_i^1 = x_i^2 = ... = x_i^p \qquad (i = 1, 2,..., n)$$

in the sense that, if $(x^1, x^2,..., x^p)$ belongs to K_n, so does $(x^j, x^{j+1},..., x^{j-1})$. Here j is any integer $< p$ and x^j is written for the row $(x_1^j, x_2^j,..., x_n^j)$.

We project K_n on to I_n by the rule

$$\phi: \qquad (x^1, x^2,..., x^p) \to (z, z,..., z),$$

where

$$z_i = \frac{1}{p} \sum_{r=1}^{p} x_i^r,$$
$$= \frac{1}{p} \sum_{r=0}^{p-1} f_i^r(x) \text{ if } x \text{ is on } K_n.$$

From the inequality

$$|z_i - x_i| = \left| \frac{1}{p} \sum_{r=0}^{p-1} (f_i^r - x_i) \right|$$
$$\leqslant \frac{1}{p} \sum_{r=0}^{p-1} \left| f_i^r - x_i \right| < \tfrac{1}{2}$$

it follows as before that the auxiliary mapping ψ of I_n on itself has degree 1 at 0, and hence that ϕ has degree ± 1. Next, projecting K_n on to the n-dimensional faces of certain p_n-cubes, after preliminary deformations which use the p different simplicial divisions

$$\{f^j(x) \times f^{j+1}(x) \times ... \times f^{j-1}(x)\} \quad (j = 1, 2,..., p)$$

of K_n, we show that the degree $\equiv 0 \pmod p$. This contradiction shows that every point of C is fixed.

In the proof of Theorem 2 there is no new point that needs explicit mention when $p > 2$.

THE RESOLVENTS OF AN EQUATION OF THE SEVENTH DEGREE

By H. O. FOULKES

[Received 1 September 1930]

1. It is well known that every irreducible equation of the seventh degree,

$$ax^7+bx^6+cx^5+dx^4+ex^3+fx^2+gx+h$$
$$\equiv a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon)(x-\zeta)(x-\eta)=0,$$

has for its Galois group a transitive substitution group of degree seven.

The substitution groups on seven letters have been tabulated by several writers,* there being forty groups in all, of which seven are transitive. These will be denoted by

$$G_{5040},\ \Gamma_{2520},\ \Gamma_{168},\ G_{42},\ \Gamma_{21},\ G_{14},\ C_7$$

where the order is equal to the suffix in each case and Γ denotes a group whose substitutions are all positive. The first three of these groups are known to be insoluble, whereas G_{42}, which contains Γ_{21}, G_{14}, and C_7, is the metacyclic group.

If $U=(\alpha\beta\gamma\delta\epsilon\zeta\eta)$, $V=(\alpha\gamma\beta\zeta\delta\epsilon)$, $W=(\beta\gamma)(\delta\zeta)$, then $\Gamma_{168}=\{U,W\}$, $G_{42}=\{U,V\}$, $\Gamma_{21}=\{U,V^2\}$, $G_{14}=\{U,V^3\}$, $C_7=\{U\}$. The group Γ_{168}, whose substitutions have been classified by Gordan,† was omitted from the lists of groups published by Askwith and Cayley, although it had been discovered thirty years before.

In considering the resolvents associated with seventh-degree equations on lines similar to those recently adopted for sextic equations‡

* E. H. Askwith, 'On possible groups of substitutions that can be formed with 3, 4, 5, 6, 7, 8 letters': *Quart. J. of Math.* 24 (1890), 111–67; A. Cayley, 'On the substitution groups for 2,..., 8 letters': *Quart. J. of Math.* 25 (1891), 71–88; *Coll. Math. Papers*, 13 (1897), 130–3; F. N. Cole, 'List of the substitution groups on 9 letters, etc.': *Quart. J. of Math.* 26 (1892), 372–88 [also *Bull. New York Math. Soc.* 2 (1893), 184–90]; G. A. Miller, 'Memoir on the substitution groups of degree $\leqslant 8$': *Amer. J. of Math.* 21 (1899), 328–37.

† P. Gordan, 'Ueber Gleichungen siebenten Grades mit einer Gruppe von 168 Substitutionen' (Part 2): *Math. Annalen*, 25 (1884), 459–521.

‡ W. E. H. Berwick, 'On soluble sextic equations': *Proc. London Math. Soc.* (2), 29 (1928–9), 1–28.

it is sufficient to deal only with those resolvents corresponding to the groups Γ_{2520}, Γ_{168}, and G_{42}. These three resolvents are of degrees 2, 30, and 120 respectively, and, although it is considered intractable to compute the 30-ic and 120-ic resolvents for the general form of a 7-ic equation containing eight coefficients, it is possible to determine the degrees and the nature of the irreducible factors of these resolvents for each of the seven types of irreducible 7-ic equation [§§ 3, 4]. It is also possible to express the roots of the 7-ic equation as rational functions of the roots of its irreducible resolvents. The form of these rational expressions is determined in § 4.

2 (*a*) If Δ is the discriminant of the equation, the function $\sqrt{\Delta}$ belongs to the alternating group Γ_{2520} and is a root of the quadratic resolvent $T^2-\Delta = 0$.

(*b*) The function*

$$\psi_0 = \gamma\alpha\delta+\delta\beta\epsilon+\epsilon\gamma\zeta+\zeta\delta\eta+\eta\epsilon\alpha+\alpha\zeta\beta+\beta\eta\gamma$$

is unaltered by U and W and takes up thirty different forms when operated on by all the substitutions of the symmetric group. The function belongs therefore to Γ_{168}, whose index is 30. Every function belonging to Γ_{168} possesses the property, observed by Noether,† of being expressible in seven 'triplets'. Kronecker,‡ in 1858, gave

$$(\gamma+\alpha+\delta)(\delta+\beta+\epsilon)(\epsilon+\gamma+\zeta)(\zeta+\delta+\eta)(\eta+\epsilon+\alpha)(\alpha+\zeta+\beta)(\beta+\eta+\gamma)$$

as a function belonging to Γ_{168}.

The thirty values of ψ_0 are the roots of an equation of degree 30 whose coefficients, being symmetric functions of $\alpha, \beta,..., \eta$, are rational in the field $[a, b,..., h]$. If we choose the suffixes so that

$$\psi_1 = (\gamma\delta)\psi_0, \qquad \psi_8 = (\delta\eta\zeta)\psi_0, \qquad \psi_{15} = (\delta\zeta\eta)\psi_0$$

$$\psi_{22} = (\gamma\zeta\eta\delta)\psi_0, \qquad \psi_{29} = (\gamma\eta\zeta\delta)\psi_0.$$

and

$$U = (\psi_0)(\psi_k\,\psi_{k+1}\,\psi_{k+2}\,\psi_{k+3}\,\psi_{k+4}\,\psi_{k+5}\,\psi_{k+6})(\psi_{29}),$$

where $$k = 1, 8, 15, 22,$$

* P. Gordan, loc. cit.

† M. Noether, 'Ueber die Gleichungen achten Grades und ihr Auftreten in der Theorie der Curven vierter Ordnung': *Math. Annalen*, 15 (1879), 89–110.

‡ L. Kronecker, 'Ueber Gleichungen des siebenten Grades', *Monatsberichte der Berlin. Acad.* April 1858.

then V and W permute the suffixes in the following way:

$$V = (0, 29)(7, 14)(21, 28)$$
$$(1, 10, 2, 13, 4, 12)(3, 9, 6, 11, 5, 8)$$
$$(15, 24, 16, 27, 18, 26)(17, 23, 20, 25, 19, 22),$$
$$W = (0)(1)(5)(7)(9)(16)$$
$$(2, 3)(4, 6)(8, 15)(10, 12)(11, 18)(13, 21)$$
$$(14, 20)(17, 19)(22, 29)(23, 25)(24, 27)(26, 28).$$

(*c*) The functions $\phi_0 = \psi_0 + \psi_{29}$ and

$$\phi' = a^4[\{(\delta-\epsilon)(\epsilon-\zeta)(\zeta-\eta)(\eta-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)\}^2+$$
$$+\{(\gamma-\zeta)(\delta-\eta)(\epsilon-\alpha)(\zeta-\beta)(\eta-\gamma)(\alpha-\delta)(\beta-\epsilon)\}^2+$$
$$+\{(\beta-\eta)(\gamma-\alpha)(\delta-\beta)(\epsilon-\gamma)(\zeta-\delta)(\eta-\epsilon)(\alpha-\zeta)\}^2]$$

are unaltered by U and V, and each takes up 120 different forms when operated on by the symmetric group. Both functions belong therefore to the metacyclic group G_{42} and either may be taken as a root of the corresponding 120-ic resolvent. In order to determine the degrees and zeros of the irreducible factors of this resolvent for the various types of 7-ic equation it is convenient to employ the simpler function ϕ_0. The 119 conjugates of ϕ_0 can be obtained from the 17 functions

$$\phi_1 = \psi_1+\psi_8, \quad \phi_{29} = \psi_{17}+\psi_{27}, \quad \phi_{57} = \psi_{15}+\psi_{29}, \quad \phi_{85} = \psi_1+\psi_{18},$$
$$\phi_8 = \psi_5+\psi_{11}, \quad \phi_{36} = \psi_{16}+\psi_{28}, \quad \phi_{64} = \psi_8+\psi_{22}, \quad \phi_{92} = \psi_8+\psi_{23},$$
$$\phi_{15} = \psi_3+\psi_{13}, \quad \phi_{43} = \psi_{19}+\psi_{25}, \quad \phi_{71} = \psi_1+\psi_{15}, \quad \phi_{99} = \psi_1+\psi_{21},$$
$$\phi_{22} = \psi_2+\psi_{14}, \quad \phi_{50} = \psi_0+\psi_{22}, \quad \phi_{78} = \psi_8+\psi_{26}, \quad \phi_{106} = \psi_8+\psi_{24},$$
$$\phi_{113} = \psi_1+\psi_{20},$$

by applying to them the substitution

$$U = (\phi_0)(\phi_k\,\phi_{k+1}\,\phi_{k+2}\,\phi_{k+3}\,\phi_{k+4}\,\phi_{k+5}\,\phi_{k+6}),$$

where $$k = 1, 8, 15, 22, \ldots, 106, 113.$$

In this notation V and W permute the suffixes as follows:

$V =$

(0)(7)(14, 21, 28)(35, 42, 49)(56, 63)(70, 77)(1, 3, 2, 6, 4, 5)
(8, 17, 23, 13, 18, 26)(9, 20, 25, 12, 15, 24)(10, 16, 27, 11, 19, 22)
(29, 38, 44, 34, 39, 47)(30, 41, 46, 33, 36, 45)(31, 37, 48, 32, 40, 43)
(50, 59, 51, 62, 53, 61)(52, 58, 55, 60, 54, 57)(64, 73, 65, 76, 67, 75)
(66, 72, 69, 74, 68, 71)(78, 115, 93, 90, 109, 103)(79, 118, 95, 89, 106, 101)
(80, 114, 97, 88, 110, 99)(81, 117, 92, 87, 107, 104)(82, 113, 94, 86, 111, 102)
(83, 116, 96, 85, 108, 100)(84, 119, 98, 91, 112, 105),

$W =$

$(0, 50)(1, 71)(2, 13)(3, 21)(4, 118)(5, 25)(6, 88)(7, 105)(8, 103)(9, 14)$
$(10, 77)(11, 113)(12, 115)(15, 114)(16, 76)(17, 89)(18, 24)(19, 27)(20, 85)$
$(22, 87)(23, 100)(26, 74)(28, 99)(29, 39)(30, 95)(31, 59)(32, 84)(33, 69)$
$(34, 92)(35, 36)(37, 61)(38, 67)(40, 112)(41, 97)(42, 106)(43, 48)(44, 70)$
$(45, 83)(46, 78)(47, 58)(49, 109)(51, 53)(52, 55)(54, 56)(57, 64)(60, 81)$
$(62, 98)(63, 111)(65, 107)(66, 96)(68, 80)(72, 101)(73, 86)(75, 117)(79, 93)$
$(82, 94)(90, 116)(91, 119)(102, 104)(108, 110).$

3. By examining at length the way in which each group permutes the ψ's and ϕ's it is possible to determine the degrees and zeros of the irreducible resolvents for each of the seven types of irreducible 7-ic equation. The following table, in which $T(\tau)$, $\Psi(\psi)$, $\Phi(\phi)$ denote respectively the quadratic, 30-ic, and 120-ic resolvents, gives the degrees of these irreducible resolvents.

	$T(\tau)$	$\Psi(\psi)$	$\Phi(\phi)$
G_{5040}	2	30	120
Γ_{2520}	1, 1	15, 15	120
Γ_{168}	1, 1	1, 7, 8, 14	8, 56, 56
G_{42}	2	2, 14, 14	1, 7, 14, 14, 21, 21, 42
Γ_{21}	1, 1	1, 1, 7, 7, 7, 7	1, 7, 7, 7, 7, 7, 21, 21, 21, 21
G_{14}	2	2, 14, 14	1, seven 7-ic, five 14-ic factors
C_7	1, 1	1, 1, 7, 7, 7, 7	1, seventeen 7-ic factors

4. The irreducible resolvents will now be examined in the light of the theorems on normal resolvents first demonstrated by Professor W. E. H. Berwick.*

(*a*) The irreducible resolvents corresponding to the symmetric and alternating groups are normal resolvents and are, of course, insoluble. They serve to indicate the existence of groups of degree greater than seven which are simply isomorphic with G_{5040} and Γ_{2520}.

(*b*) The 7-ic resolvent corresponding to Γ_{168} has a group generated by

$$U = (\psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \psi_7), \qquad W = (\psi_2 \psi_3)(\psi_4 \psi_6),$$

which is Γ_{168} itself. In this case the permutations which leave α unaltered also leave ψ_1 unaltered, so that $\alpha = R(\psi_1)$, where R is a rational function. Also, by applying U, we find

$$\beta = R(\psi_2), \ \ \gamma = R(\psi_3), \ \ \delta = R(\psi_4), \ \ \epsilon = R(\psi_5), \ \ \zeta = R(\psi_6), \ \ \eta = R(\psi_7).$$

* W. E. H. Berwick, loc. cit., 12–14.

The two octavic resolvents of Γ_{168} 7-ic equations are also normal resolvents, since

$$U = (\psi_{22}\psi_{23}\psi_{24}\psi_{25}\psi_{26}\psi_{27}\psi_{28}),$$
$$W = (\psi_{22}\psi_{29})(\psi_{23}\psi_{25})(\psi_{24}\psi_{27})(\psi_{26}\psi_{28})$$

are found to generate a group of degree eight simply isomorphic with Γ_{168}. The permutations of Γ_{168} which do not alter α form a subgroup of degree 6 and order 24 generated by

$$(\beta\delta\zeta\gamma)(\epsilon\eta) = (\psi_{22}\psi_{23}\psi_{28}\psi_{24})(\psi_{25}\psi_{29}\psi_{27}\psi_{26}),$$
$$(\beta\epsilon\gamma)(\delta\zeta\eta) = (\psi_{23}\psi_{26}\psi_{24})(\psi_{25}\psi_{27}\psi_{28}),$$
$$(\beta\gamma)(\delta\zeta) = (\psi_{22}\psi_{29})(\psi_{23}\psi_{25})(\psi_{24}\psi_{27})(\psi_{26}\psi_{28}).$$

This group of degree 6 has been denoted by Γ_{24} by Professor Berwick, and the corresponding group of order 24 on the eight ψ's appears as 24_{11} in Miller's* list of groups of degree 8. Both groups are simply isomorphic with H_{24}, the symmetric group on four letters. If F denotes a rational function unaltered in form by this group of degree 8 and order 24, then $\alpha = F(\psi_{22}, \psi_{23}, \psi_{24}, \psi_{25}, \psi_{26}, \psi_{27}, \psi_{28}, \psi_{29})$ and $\beta = U(\alpha), \gamma = U^2(\alpha),\ldots, \eta = U^6(\alpha)$.

Equivalent expressions for the roots in terms of the roots of the octavic ϕ-resolvent can be obtained by putting

$$\psi_{29} = \phi_0, \qquad \psi_{21+i} = \phi_{49+i} \qquad [i = 1, 2,\ldots, 7]$$

in the above.

(c) The single 7-ic resolvent of a G_{42} 7-ic equation is easily seen to be itself a G_{42} equation since its group is generated by

$$U = (\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\phi_7), \qquad V = (\phi_1\phi_3\phi_2\phi_6\phi_4\phi_5).$$

The powers of the cycle

$$U^4V = (\beta\delta\gamma\eta\epsilon\zeta) = (\phi_2\phi_4\phi_3\phi_7\phi_5\phi_6)$$

form a group of order 6 which leaves α unaltered, so that

$$\alpha = R(\phi_1) \quad \text{or} \quad \alpha = C(\phi_2, \phi_4, \phi_3, \phi_7, \phi_5, \phi_6),$$

where R denotes a rational function and C a rational cyclical function. As before, $\beta = U(\alpha)$, $\gamma = U^2(\alpha)$, etc.

Each 14-ic resolvent is normal, the group being generated in each case by permutations of the form

$$U = (\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6\psi_7)(\psi_8\psi_9\psi_{10}\psi_{11}\psi_{12}\psi_{13}\psi_{14}),$$
$$V = (\psi_1\psi_{10}\psi_2\psi_{13}\psi_4\psi_{12})(\psi_3\psi_9\psi_6\psi_{11}\psi_5\psi_8)(\psi_7\psi_{14}).$$

This group is imprimitive in two ways, so that this 14-ic resolvent

* G. A. Miller, loc. cit.

belongs to a class of equations which factorize into two Γ_{21} 7-ic factors in a certain quadratic field and into seven quadratic factors in a certain Γ_{21} 7-ic field. By the adjunction of $\sqrt{\Delta}$ to the field of rationality, the group of the 7-ic equation is reduced from G_{42} to Γ_{21}, and it is in this enlarged field that the 14-ic resolvent breaks up into two Γ_{21} 7-ic factors. The permutation

$$U^4V = (\beta\delta\gamma\eta\epsilon\zeta) = (\psi_1\psi_8)(\psi_2\psi_{11}\psi_3\psi_{14}\psi_5\psi_{13})(\psi_4\psi_{10}\psi_7\psi_{12}\psi_6\psi_9)$$

and its powers leave α unaltered, so that α is expressible as a rational function of the fourteen ψ's unaltered in form by this permutation. Hence $\alpha = S(\psi_1, \psi_8)$ where S is a symmetric function. This agrees with our previous result, $\alpha = R(\phi_1)$, since $\phi_1 = \psi_1 + \psi_8$ and can be taken to be any symmetric function of ψ_1 and ψ_8. Similarly,

$$\beta = S(\psi_2, \psi_9), \qquad \gamma = S(\psi_3, \psi_{10}), \qquad \delta = S(\psi_4, \psi_{11}),$$
$$\epsilon = S(\psi_5, \psi_{12}), \qquad \zeta = S(\psi_6, \psi_{13}), \qquad \eta = S(\psi_7, \psi_{14}).$$

Equivalent expressions for the roots in terms of the roots of the other 14-ic resolvents can be obtained by replacing ψ_i by ψ_{14+i}, ϕ_{49+i} or ϕ_{63+i} in the above formulae.

The groups of the two 21-ic resolvents are generated by permutations of the form

$$U = (\phi_8\phi_9\phi_{10}\phi_{11}\phi_{12}\phi_{13}\phi_{14})(\phi_{15}\phi_{16}\phi_{17}\phi_{18}\phi_{19}\phi_{20}\phi_{21}) \times \\ \times (\phi_{22}\phi_{23}\phi_{24}\phi_{25}\phi_{26}\phi_{27}\phi_{28}),$$

$$V = (\phi_8\phi_{17}\phi_{23}\phi_{13}\phi_{18}\phi_{26})(\phi_9\phi_{20}\phi_{25}\phi_{12}\phi_{15}\phi_{24}) \times \\ \times (\phi_{10}\phi_{16}\phi_{27}\phi_{11}\phi_{19}\phi_{22})(\phi_{14}\phi_{21}\phi_{28}),$$

which give rise to a group of degree 21 and order 42 imprimitive in two ways. These resolvents are therefore normal and belong to a class of equations of degree 21 which factorize into three G_{14} 7-ic factors in a normal cubic field and also into seven normal cubics in a G_{14} 7-ic field. A suitable cubic irrationality is given in § 5; the adjunction of a root of this cubic lowers the group of the 7-ic equation from G_{42} to G_{14} and splits up the 21-ic resolvent into three G_{14} 7-ic factors. The permutation

$$U^4V = (\phi_8\phi_{15}\phi_{22})(\phi_9\phi_{18}\phi_{24}\phi_{14}\phi_{19}\phi_{27})(\phi_{10}\phi_{21}\phi_{26}\phi_{13}\phi_{16}\phi_{25}) \times \\ \times (\phi_{11}\phi_{17}\phi_{28}\phi_{12}\phi_{20}\phi_{23})$$

leaves α unaltered, so that α is expressible as a rational function of the ϕ's unaltered in form by U^4V, e.g. $\alpha = C(\phi_8, \phi_{15}, \phi_{22})$, where C is

a cyclical function, since α and this function are unaltered by the same six permutations of G_{42}. Similarly

$$\beta = C(\phi_9, \phi_{16}, \phi_{23}), \qquad \epsilon = C(\phi_{12}, \phi_{19}, \phi_{26}),$$
$$\gamma = C(\phi_{10}, \phi_{17}, \phi_{24}), \qquad \zeta = C(\phi_{13}, \phi_{20}, \phi_{27}),$$
$$\delta = C(\phi_{11}, \phi_{18}, \phi_{25}), \qquad \eta = C(\phi_{14}, \phi_{21}, \phi_{28}).$$

We obtain similar formulae for the other 21-ic ϕ-resolvent by replacing ϕ_i by ϕ_{21+i}.

Finally, the resolvent of degree 42

$$(\phi-\phi_{78})(\phi-\phi_{79})\ldots(\phi-\phi_{118})(\phi-\phi_{119})$$

is a normal resolvent, its group being imprimitive in six ways. It is found that this resolvent can be factorized into

(i) six cyclic 7-ics in a cyclic sextic field,
(ii) seven cyclic sextics in a cyclic 7-ic field,
(iii) three 14-ic factors in an alternating cubic field,
(iv) fourteen alternating cubics in a certain 14-ic field,
(v) two 21-ic factors in a quadratic field,
(vi) twenty-one quadratics in a certain 21-ic field.

The quadratic field arising in (v) is $[\sqrt{\Delta}]$, and the sextic and cubic fields in (i) and (iii) respectively are given in § 5.

(*d*) Each of the nine irreducible resolvents of degree 7 corresponding to Γ_{21} has a group generated by permutations of the form

$$U = (\psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \psi_7), \qquad V^2 = (\psi_1 \psi_2 \psi_4)(\psi_3 \psi_6 \psi_5),$$

which is Γ_{21} itself. The sub-group leaving α unaltered consists of the powers of

$$U^3V^2 = (\beta\gamma\epsilon)(\delta\eta\zeta) = (\psi_2 \psi_3 \psi_5)(\psi_4 \psi_7 \psi_6),$$

so that $\alpha = R(\psi_1)$ and $\alpha = C(\psi_2, \psi_3, \psi_5)$, where both functions are rational and C is cyclical. By applying the permutation U we obtain expressions for β, γ, δ, ϵ, ζ, η. We obtain similar formulae involving the roots of the other 7-ic resolvents by replacing ψ_i by any of the functions

$$\psi_{7+i}, \quad \psi_{14+i}, \quad \psi_{21+i}, \quad \phi_i, \quad \phi_{49+i}, \quad \phi_{56+i}, \quad \phi_{63+i}, \quad \phi_{70+i}.$$

Each of the four irreducible 21-ic resolvents of a Γ_{21} 7-ic equation has a group of degree 21 generated by permutations of the form

$$U = (\phi_8 \phi_9 \phi_{10} \phi_{11} \phi_{12} \phi_{13} \phi_{14})(\phi_{15} \phi_{16} \phi_{17} \phi_{18} \phi_{19} \phi_{20} \phi_{21}) \times$$
$$\times (\phi_{22} \phi_{23} \phi_{24} \phi_{25} \phi_{26} \phi_{27} \phi_{28}),$$
$$V^2 = (\phi_8 \phi_{23} \phi_{18})(\phi_9 \phi_{25} \phi_{15})(\phi_{10} \phi_{27} \phi_{19})(\phi_{11} \phi_{22} \phi_{16})(\phi_{12} \phi_{24} \phi_{20}) \times$$
$$\times (\phi_{13} \phi_{26} \phi_{17})(\phi_{14} \phi_{28} \phi_{21}).$$

These resolvents belong to a class of equations which factorize into three cyclic 7-ic factors in an alternating cubic field and into seven alternating cubics in a cyclic 7-ic field. The cubic irrationality in the former case reduces the group of the 7-ic equation from Γ_{21} to C_7 when it is adjoined to the field $[a, b, ..., h]$. The permutations of Γ_{21} which do not alter α are the powers of

$$U^3V^2 = (\phi_8\phi_{22}\phi_{15})(\phi_9\phi_{24}\phi_{19})(\phi_{10}\phi_{26}\phi_{16})(\phi_{11}\phi_{28}\phi_{20})(\phi_{12}\phi_{23}\phi_{17})\times$$
$$\times(\phi_{13}\phi_{25}\phi_{21})(\phi_{14}\phi_{27}\phi_{18}),$$

so that $\alpha = C(\phi_8, \phi_{15}, \phi_{22})$, a cyclical function. The expressions for $\beta, \gamma, ..., \eta$ are the same as in (*c*). Equivalent formulae for the roots may be obtained by replacing ϕ_i by ϕ_{21+i}, or by

$$\begin{array}{llll} \text{(i) } \phi_{70+i}, & i = 8,..., 14, & \text{(ii) } \phi_{77+i}, & i = 8,..., 14, \\ \phi_{77+i}, & i = 15,..., 21, & \phi_{84+i}, & i = 15,..., 21, \\ \phi_{84+i}, & i = 22,..., 28; & \phi_{91+i}, & i = 22,..., 28; \end{array}$$

the four sets of ϕ's corresponding to the four irreducible 21-ic resolvents.

(*e*) The group associated with the 7-ic resolvents of a G_{14} 7-ic equation is generated by

$$U = (\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\phi_7), \qquad V^3 = (\phi_1\phi_6)(\phi_2\phi_5)(\phi_3\phi_4),$$

which is G_{14} itself. The two permutations of G_{14} which do not alter α are 1 and

$$U^5V^3 = (\beta\eta)(\gamma\zeta)(\delta\epsilon) = (\phi_2\phi_7)(\phi_3\phi_6)(\phi_4\phi_5).$$

Hence

$$\begin{aligned}
\alpha &= R(\phi_1) = S_1(\phi_2, \phi_7) = S_2(\phi_3, \phi_6) = S_3(\phi_4, \phi_5),\\
\beta &= R(\phi_2) = S_1(\phi_3, \phi_1) = S_2(\phi_4, \phi_7) = S_3(\phi_5, \phi_6),\\
\gamma &= R(\phi_3) = S_1(\phi_4, \phi_2) = S_2(\phi_5, \phi_1) = S_3(\phi_6, \phi_7),\\
\delta &= R(\phi_4) = S_1(\phi_5, \phi_3) = S_2(\phi_6, \phi_2) = S_3(\phi_7, \phi_1),\\
\epsilon &= R(\phi_5) = S_1(\phi_6, \phi_4) = S_2(\phi_7, \phi_3) = S_3(\phi_1, \phi_2),\\
\zeta &= R(\phi_6) = S_1(\phi_7, \phi_5) = S_2(\phi_1, \phi_4) = S_3(\phi_2, \phi_3),\\
\eta &= R(\phi_7) = S_1(\phi_1, \phi_6) = S_2(\phi_2, \phi_5) = S_3(\phi_3, \phi_4),
\end{aligned}$$

where R denotes a rational function and S_r a symmetric function. We may replace ϕ_i in the above by

$$\phi_{7+i}, \quad \phi_{14+i}, \quad \phi_{21+i}, \quad \phi_{28+i}, \quad \phi_{35+i}, \quad \phi_{42+i}$$

successively, so as to obtain expressions for $\alpha, \beta, ..., \eta$ in terms of the roots of each of the 7-ic resolvents.

There are seven 14-ic resolvents of a G_{14} 7-ic equation, their group being generated by permutations of the form

$$U = (\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6\psi_7)(\psi_8\psi_9\psi_{10}\psi_{11}\psi_{12}\psi_{13}\psi_{14}),$$
$$V^3 = (\psi_1\psi_{13})(\psi_2\psi_{12})(\psi_3\psi_{11})(\psi_4\psi_{10})(\psi_5\psi_9)(\psi_6\psi_8)(\psi_7\psi_{14}).$$

This group is of order 14 and, being imprimitive in two ways, gives rise to a class of equations which factorize into seven quadratics in a cyclic 7-ic field and into two cyclic 7-ic factors in the field $[\sqrt{\Delta}]$. The rational expression for α in terms of the fourteen ψ's is unaltered in form by

$$U^5V^3 = (\psi_1\psi_8)(\psi_2\psi_{14})(\psi_3\psi_{13})(\psi_4\psi_{12})(\psi_5\psi_{11})(\psi_6\psi_{10})(\psi_7\psi_9),$$

i.e. $\alpha = S_1(\psi_1, \psi_8) = S_2(\psi_2, \psi_{14}) = S_3(\psi_3, \psi_{16})$, etc.,

formulae which are equivalent to

$$\alpha = R(\phi_1) = R_1(\phi_{22}) = R_2(\phi_{15}), \text{ etc.},$$

already obtained, since $\phi_1 = \psi_1+\psi_8$, etc. By applying the cyclic permutation U we can obtain similar expressions for $\beta, \gamma,..., \eta$; also, by replacing ψ_i in the above by any of the sets of functions

$$\psi_{14+i}, \quad \phi_{49+i}, \quad \phi_{63+i}, \quad \phi_{77+i}, \quad \phi_{91+i}, \quad \phi_{105+i},$$

we obtain expressions for the seven roots of the equation in terms of the roots of any of its 14-ic resolvents.

(*f*) The twenty-one 7-ic resolvents of a cyclic 7-ic equation are easily seen to be themselves cyclical, since the group is generated by the cycle $U = (\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6\psi_7)$. These resolvents are therefore normal resolvents, being in fact Tschirnhausen transformations of the equation and of each other.

5. It is theoretically possible, by means of the three resolvents considered here, to ascertain the group of any irreducible equation of degree 7, though the method appears to be of no practical use owing to the high degrees of two of the resolvents. The function ϕ' in § 2 (*c*) will be a root of a resolvent of degree 120 whose coefficients are invariants of the binary form

$$ax^7+bx^6y+cx^5y^2+dx^4y^3+ex^3y^4+fx^2y^5+gxy^6+hy^7,$$

but the method of computing these coefficients, though possible for $n = 5$ and $n = 6$,* is too laborious for $n = 7$. Runge† and

* W. E. H. Berwick, loc. cit.; also 'The condition that a quintic equation should be soluble by radicals': *Proc. London Math. Soc.* (2), 14 (1915), 301–7.

† C. Runge, 'Über die auflösbaren Gleichungen von der Form

$$x^5+ux+v = 0\text{'}:$$

Acta Math. 7 (1885), 173–86.

Wäisälä* have respectively computed resolvents for the quintic and sextic, using the trinomial form with only two coefficients, and some such simplification appears to be necessary to compute the 30-ic and 120-ic resolvents of a 7-ic equation.

When the three resolvents are irreducible, the group of the given equation is the symmetric group, and reduces to the alternating group when the quadratic resolvent has a rational root. If the 30-ic resolvent has a rational root but the 120-ic resolvent has not, then the equation has Γ_{168} as its group.

When the 120-ic resolvent has a rational root, the equation is soluble by radicals, its group being G_{42}, Γ_{21}, G_{14}, or C_7. It is theoretically possible to distinguish between these groups by means of the degrees of the irreducible factors of the 120-ic resolvent, but the problem may be made to depend on a certain cubic equation in the following way.

If we take the invariant function

$$\phi' = z_1+z_2+z_3,$$

where

$$z_1 = a^4\{(\delta-\epsilon)(\epsilon-\zeta)(\zeta-\eta)(\eta-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)\}^2,$$

$$z_2 = a^4\{(\gamma-\zeta)(\delta-\eta)(\epsilon-\alpha)(\zeta-\beta)(\eta-\gamma)(\alpha-\delta)(\beta-\epsilon)\}^2,$$

$$z_3 = a^4\{(\beta-\eta)(\gamma-\alpha)(\delta-\beta)(\epsilon-\gamma)(\zeta-\delta)(\eta-\epsilon)(\alpha-\zeta)\}^2,$$

as a root of the 120-ic resolvent, then ϕ' is rational for all soluble 7-ics and z_1, z_2, z_3 are the roots of an alternating cubic

$$z^3-\phi' z^2+Az-\Delta = 0,$$

where Δ is the discriminant of the 7-ic equation and A is a metacyclic function, being thus a rational function of ϕ', and is itself the rational root of another 120-ic resolvent. If Δ is not a rational square, the group is G_{42} or G_{14} according as the cubic does, or does not, remain irreducible. If Δ is a rational square, then the group is Γ_{21} if the cubic is irreducible, and C_7 if the cubic has three rational linear factors.

Another method, equivalent to the preceding, is to consider the cyclic sextic

$$z^6-\phi' z^4+Az^2-\Delta = 0.$$

* K. Wäisälä, 'Über die algebraisch auflösbaren Gleichungen sechsten Grades': *Ann. Acad. Sc. Fennicae*, A (13) (1916).

The correspondence between the groups G_{42}, Γ_{21}, G_{14}, C_7, and the different forms of this sextic is shown in the following table:

7-ic	Sextic
G_{42}	Cyclic
Γ_{21}	Two normal cubics
G_{14}	Three quadratics
C_7	Six linear factors

6. Equations whose groups are G_{5040}, Γ_{2520}, or Γ_{168} are not soluble by radicals, but other solutions have been obtained.* G_{42}, Γ_{21}, G_{14}, and C_7 equations are soluble by radicals, the solution involving a cyclic sextic or some modification of it. In order to find all equations having the Galois group G_{42}, i.e. to construct an algebraic expression which shall always be a root of a G_{42} 7-ic, it is necessary to find a general algebraic expression for a root of a cyclic sextic, a problem which has been worked out by Breuer.†

In the presentation of these results I am greatly indebted to Professor W. E. H. Berwick, who read the paper in manuscript and who has made many suggestions for its improvement.

* F. Klein, *Ges. Math. Abhand.* 2, 388–439; Radford, 'On the solution of certain equations of the seventh degree': *Quart. J. of Math.*, 30 (1898–9), 263–306.

† Breuer, 'Über cyklische Gleichungen und Minimalbasis': *Math. Annalen*, 86 (1922), 108–13.

SOME RESULTS IN THE THEORY OF CONFORMAL REPRESENTATION

By J. HODGKINSON

[Received 31 October 1930]

SUPPOSE we take, in the plane of a complex variable u, a triangle bounded by circular arcs and having angles $\pi/2$, $\pi/3$, π/n, where n is a positive integer, and also, in the plane of a variable τ, the fundamental triangle of the elliptic modular configuration, which has angles $\pi/2$, $\pi/3$, 0. Suppose, further, that both these triangles are mapped upon the same half-plane of a variable z. If, from the two relations effecting these representations, the intermediate variable z is eliminated, u is a one-valued function of τ. Explicit evaluations of u were given by Klein,* who announced this theorem, for $n = 2, 3, 4, 5$. In other words, modular expressions were obtained for the *dihedral*, *tetrahedral*, *octahedral*, and *icosahedral irrationalities*, in Klein's phraseology.

I give below an evaluation of u when $n = 6$. For reasons assigned later, I have called the function u the *transcendant of the equilateral triangle*.

I obtain also an expression for the *transcendant of the isosceles right-angled triangle*. At sight it appeared necessary first to obtain the representation of a triangle, in the plane of τ, with angles $\pi/2$, $\pi/4$, 0. This, in fact, proved unnecessary. But this triangle is one of a set of triangles which have been enumerated as having the following property. Suppose two triangles of this set in the plane of the same variable τ are mapped upon the half-planes of variables z_1, z_2; then there is an algebraical relation between z_1 and z_2 as functions of τ.† I have obtained the functions of τ thus required for all the triangles of the set (in brief, the *triangle-functions*), except that, in one instance, I am not at present able to exhibit the function, as the theory demands, as a one-valued function of τ. For this purpose I rely upon a method given by Burnside for the representation of a figure composed of Schwarzian repetitions of a simpler figure.‡

It is part of the general theory that the triangle-function is fundamental for its appropriate group of linear substitutions, i.e. every

* F. Klein, *Vorlesungen über das Ikosaeder* (1884), 132.

† J. Hodgkinson, *Proc. London Math. Soc.* (2) 18 (1919), 268–73.

‡ W. Burnside, *Proc. London Math. Soc.* (1) 24 (1893), 187–206, and especially 194–7.

one-valued function automorphic for that group is a one-valued function of the triangle-function. Some of the groups so obtained are not included in the elliptic modular group as sub-groups, though, of course, they are closely allied to that group.

An intermediate result has some interest, if new. The functions

$$\{1-(4cc')^{\frac{1}{3}}\}^{\frac{1}{2}},\ \{1-\omega(4cc')^{\frac{1}{3}}\}^{\frac{1}{2}},\ \{1-\omega^2(4cc')^{\frac{1}{3}}\}^{\frac{1}{2}}$$

—where $c=k^2$, $c'=k'^2=1-k^2$, in the Jacobian theory of elliptic functions, and ω, ω^2 are cube roots of unity—are shown to be one-valued functions of iK'/K.

1. The transcendant of the equilateral triangle. The Schwarz-Christoffel formula gives the function of the (rectilinear) triangle of angles $\pi/2$, $\pi/3$, $\pi/6$ in the form

$$\frac{du}{dz}=\frac{C}{z^{\frac{2}{3}}(z-1)^{\frac{1}{2}}},$$

the correspondence of points being as indicated in the figure.

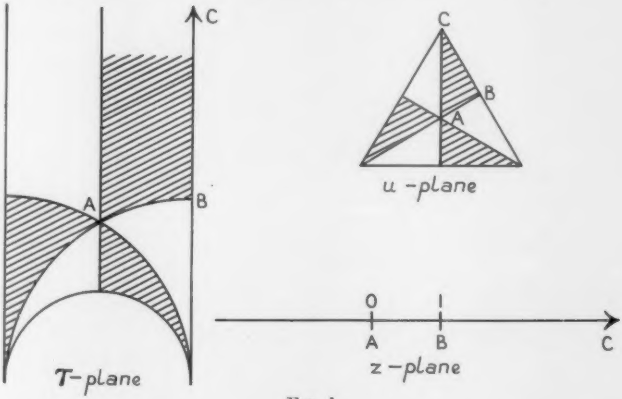


FIG. 1.

The function of the fundamental triangle of the elliptic modular configuration is*

$$\tau=\frac{iK'}{K}; \qquad z=\frac{4}{27}\,\frac{(1-cc')^3}{c^2c'^2},$$

in the Jacobian notation, whence

$$\frac{du}{dc}=\frac{-3^{\frac{1}{2}}2^{\frac{2}{3}}C}{c^{\frac{2}{3}}c'^{\frac{2}{3}}}.$$

* Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, 1 (1890), 69.

We note that this is the formula giving the representation of an equilateral triangle on the half-plane of the variable c. This is an obvious intermediate step, for six repetitions of a triangle of angles $\pi/2$, $\pi/3$, $\pi/6$ build up an equilateral triangle, and the six corresponding repetitions of the triangle of angles $\pi/2$, $\pi/3$, 0 build up a triangle of three zero angles, and the function of this triangle is c itself. For this reason u, when evaluated, can be taken as an expression of the transcendant of the equilateral triangle.

Now * $$c=16q\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^{8}, \qquad (q=e^{i\pi\tau});$$

so that $$\frac{1}{c}\frac{dc}{dq}=\frac{1}{q}\left\{1+8\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1+q^n}\right\}.$$

But $$\frac{k'^2\operatorname{sn}(2Kx/\pi)}{\operatorname{dn}(2Kx/\pi)+\operatorname{cn}(2Kx/\pi)}=\frac{\operatorname{dn}(2Kx/\pi)-\operatorname{cn}(2Kx/\pi)}{\operatorname{sn}(2Kx/\pi)}$$

$$=\frac{\pi}{2K}\frac{\sin x}{1+\cos x}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{(-1)^n q^n\sin nx}{1+q^n}.$$

Dividing by x and letting x tend to zero, we get

$$\frac{Kk'^2}{\pi}=\frac{\pi}{4K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1+q^n}.$$

Hence $\dfrac{1}{c}\dfrac{dc}{dq}=\dfrac{4K^2c'}{\pi^2 q}$,

$$\frac{du}{dq}=\frac{-3^{\frac{1}{2}}2^{\frac{8}{3}}CK^2(cc')^{\frac{1}{3}}}{\pi^2 q}=-3^{\frac{1}{2}}4Cq^{-\frac{2}{3}}\prod_{n=1}^{\infty}(1-q^{2n})^4.$$

But $$\prod_{n=1}^{\infty}(1-q^{2n})=\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-n};$$

and $$\prod_{n=1}^{\infty}(1-q^{2n})^3=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^{n-1}(2n-1)q^{n^2-n};$$

whence, assigning a proper value to C,

$$u=\sum_{m,n=-\infty}^{\infty}\frac{(-1)^{m+n}(2n-1)q^{3(m-\frac{1}{6})^2+(n-\frac{1}{2})^2}}{3(m-\frac{1}{6})^2+(n-\frac{1}{2})^2}.$$

This is the evaluation desired.

* Classical formulae such as this are quoted without reference.

From the Schwarzian theory of the representation of a curvilinear triangle* we know that

$$u = A\frac{c^{\frac{1}{3}}F(1,\frac{2}{3};\frac{4}{3};c)}{F(\frac{2}{3},\frac{1}{3};\frac{2}{3};c)} = Ac^{\frac{1}{3}}(1-c)^{\frac{1}{3}}F(1,\frac{2}{3};\frac{4}{3};c).$$

From this we readily deduce that

$$F(1,\tfrac{2}{3};\tfrac{4}{3};c)$$
$$= \tfrac{1}{6}q^{-\frac{1}{3}}\prod_{r=1}^{\infty}(1+q^{2r-1})^{8}\sum_{m,n=-\infty}^{\infty}\frac{(-1)^{m+n-1}(2n-1)q^{3(m-\frac{1}{6})^2+(n-\frac{1}{2})^2}}{3(m-\frac{1}{6})^2+(n-\frac{1}{2})^2}.$$

2. The transcendant of the isosceles right-angled triangle. As has been already observed, procedure strictly analogous to that

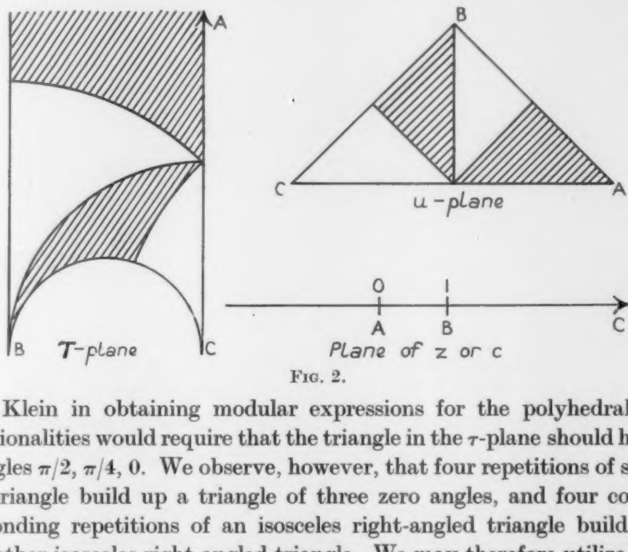


Fig. 2.

of Klein in obtaining modular expressions for the polyhedral irrationalities would require that the triangle in the τ-plane should have angles $\pi/2$, $\pi/4$, 0. We observe, however, that four repetitions of such a triangle build up a triangle of three zero angles, and four corresponding repetitions of an isosceles right-angled triangle build up another isosceles right-angled triangle. We may therefore utilize the function of the triangle of three zero angles at once, i.e. $z = c$.

The function of the isosceles right-angled triangle is given by

$$\frac{du}{dz} = \frac{C}{z^{\frac{3}{4}}(z-1)^{\frac{1}{2}}}.$$

Substituting $z = c$, we get

$$\frac{du}{dq} = 2Cq^{-\frac{3}{4}}\prod_{n=1}^{\infty}(1-q^{n})^{2}\prod_{n=1}^{\infty}(1-q^{2n})^{2}.$$

* H. A. Schwarz, *Journal für Math.* 75 (1872), 292–335.

Using again the identity

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(3n^2-n)},$$

and assigning a suitable value to C, we find that

$$u=\sum_{n_r=-\infty}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}q^{\frac{3}{2}(n_1-\frac{1}{6})^2+\frac{3}{2}(n_2-\frac{1}{6})^2+3(n_3-\frac{1}{6})^2+3(n_4-\frac{1}{6})^2}}{\frac{3}{2}(n_1-\frac{1}{6})^2+\frac{3}{2}(n_2-\frac{1}{6})^2+3(n_3-\frac{1}{6})^2+3(n_4-\frac{1}{6})^2}.$$

This leads, by an argument analogous to that of the last section, to a modular expression for $F(1,\frac{1}{2};\frac{1}{4};c)$. An expression so complicated, however, would appear to have little interest.

3. The functions of the triangles of angles $\pi/2$, 0, 0, and $\pi/3$, $\pi/3$, 0. In addition to the (already-quoted) functions of the triangles of angles $\pi/2$, $\pi/3$, 0 and 0, 0, 0,

i.e. $$\frac{4(1-cc')^3}{27c^2c'^2} \quad\text{and}\quad c,$$

the function of the triangle of angles $\pi/2$, 0, 0 is well known. It is, in fact, $c^2/(2-c)^2$. Since two repetitions build up a triangle of zero angles (see Fig. 1), it is most easily calculated by Burnside's method from the function of that triangle. Its group of substitutions is generated by (i) $\tau'=\tau+1$, (ii) $\tau'=\tau/(1-2\tau)$.

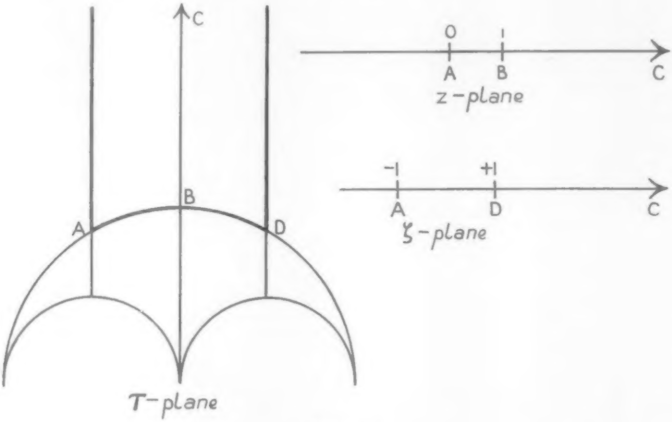


Fig. 3.

The composition of a triangle of angles $\pi/3$, $\pi/3$, 0 is shown in the figure above.

In mapping this on the half-plane of a variable ζ, it is convenient on account of the symmetry of the figure to make the vertices of the angles $\pi/3$ to correspond to the points ± 1. Then z, as a function of ζ, has a zero at the points ± 1 and a double pole at infinity, while $z-1$ has a double zero (by symmetry) at $\zeta = 0$,

$$\text{i.e.}\qquad z = 1-\zeta^2 \quad\text{or}\quad \zeta = (1-z)^{\frac{1}{2}} = \frac{i(1+c)(1-2c)(2-c)}{3\sqrt{3cc'}}.$$

Generating substitutions of its group are found, according to the Schwarzian theory, by successive inversions in two of the sides of the triangle. A pair is (i) $\tau' = \tau+2$, (ii) $\tau' = -(\tau+1)/\tau$.

4. **The function of the triangle of angles $\pi/3$, 0, 0.** A triangle of angles $\pi/3$, 0, 0 can be built up of four repetitions of a triangle of angles $\pi/2$, $\pi/3$, 0 (see Fig. 4).

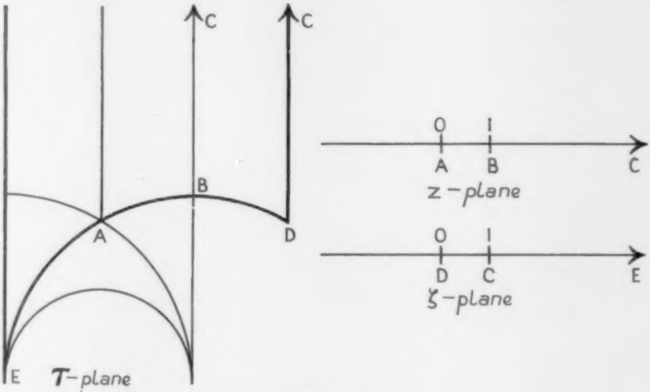


FIG. 4.

If we map the triangle upon a ζ half-plane we know that the relation between z, ζ is of the first degree in z and the fourth in ζ. Now z, *qua* function of ζ, has a simple zero at $\zeta = 0$, a triple zero at $\zeta = -\alpha$, where α is positive and is to be determined, a triple pole at $\zeta = 1$, and a simple pole at infinity, while $z-1$ has two real double zeros.

Writing

$$z = \frac{\zeta(\zeta+\alpha)^3}{\lambda^2(\zeta-1)^3}$$

we find that $\alpha = -\lambda = 8$. We have thus to solve the biquadratic in ζ,

$$\zeta^4+8(3-8z)\zeta^3+192(1+z)\zeta^2+64(8-3z)\zeta+64z = 0.$$

This leads to

$$\zeta = 2[(1+4z^{\frac{1}{3}}-8z^{\frac{2}{3}})(1+z^{\frac{1}{3}}+z^{\frac{2}{3}})^{\frac{1}{2}}+(1+4\omega z^{\frac{1}{3}}-8\omega^2 z^{\frac{2}{3}})(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})^{\frac{1}{2}}+$$
$$+(1+4\omega^2 z^{\frac{1}{3}}-8\omega z^{\frac{2}{3}})(1+\omega^2 z^{\frac{1}{3}}+\omega z^{\frac{2}{3}})^{\frac{1}{2}}-(3-8z)],$$

where ω is a primitive cube root of unity; the signs attached to the radicals are determined, when $|z|$ is small, by the fact that ζ vanishes when z vanishes. It is desirable, however, to exhibit this function in a form which is obviously a one-valued function of τ.

It proves necessary to determine the signs to be given to the radicals when z is large and positive.

As ζ increases from 0 to 1, z diminishes from 0 to infinity in the negative direction. The path of the z-point, corresponding to a small semicircle about $\zeta = 1$ in the upper half of the ζ-plane, and thus described clockwise, is a circle and a half of large radius described counter-clockwise, i.e. arg z increases by 3π. This brings us to the large positive value of z corresponding to a value of ζ slightly greater than 1.

As z moves from 0 to a large negative number, $\arg(1+z^{\frac{1}{3}}+z^{\frac{2}{3}})^{\frac{1}{2}}$ is zero throughout. As z, describing the large circle, increases its argument by 3π, $\arg(1+z^{\frac{1}{3}}+z^{\frac{2}{3}})^{\frac{1}{2}}$ is increased to π. Hence the approximate value of $(1+z^{\frac{1}{3}}+z^{\frac{2}{3}})^{\frac{1}{2}}$, when z is large and positive, is $-z^{\frac{1}{3}}$.

Again, at $z = 0$, $\arg(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})^{\frac{1}{2}}$ is zero. On passing through the point $z = -8$, $R(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})$ vanishes and changes sign, while $I(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})$ remains negative throughout. Consequently, when z is large and negative, $\arg(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})$ is $-2\pi/3$, i.e. $\arg(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})^{\frac{1}{2}}$ is $-\pi/3$. Thus the final value of $\arg(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})^{\frac{1}{2}}$ is $2\pi/3$, i.e. the approximate value of $(1+\omega z^{\frac{1}{3}}+\omega^2 z^{\frac{2}{3}})^{\frac{1}{2}}$ is $\omega z^{\frac{1}{3}}$.

Finally, $\arg(1+\omega^2 z^{\frac{1}{3}}+\omega z^{\frac{2}{3}})^{\frac{1}{2}}$ increases from 0 to $4\pi/3$, and the approximate value of $(1+\omega^2 z^{\frac{1}{3}}+\omega z^{\frac{2}{3}})^{\frac{1}{2}}$ is $\omega^2 z^{\frac{1}{3}}$.

Hence the value of ζ suitable for expansion in descending powers of $z^{\frac{1}{3}}$ is

$$2[(8z^{\frac{2}{3}}-4z^{\frac{1}{3}}-1)(z^{\frac{2}{3}}-z^{\frac{1}{3}}+1)^{\frac{1}{2}}-(8\omega^2 z^{\frac{2}{3}}-4\omega z^{\frac{1}{3}}-1)(\omega^2 z^{\frac{2}{3}}+\omega z^{\frac{1}{3}}+1)^{\frac{1}{2}}-$$
$$-(8\omega z^{\frac{2}{3}}-4\omega^2 z^{\frac{1}{3}}-1)(\omega z^{\frac{2}{3}}+\omega^2 z^{\frac{1}{3}}+1)^{\frac{1}{2}}+8z-3].$$

This change of form has been necessary in order that we may identify the expressions now to be found with the corresponding expressions here, for z is large when c is small.

Now $$z^{\frac{1}{3}}-1 = \frac{\{2^{\frac{1}{3}}+(cc')^{\frac{1}{3}}\}^2\{1-(4cc')^{\frac{1}{3}}\}}{3(cc')^{\frac{2}{3}}}.$$

Consider next the equation

$$\operatorname{cn} 2u = \operatorname{cn} u.$$

Putting $\operatorname{cn} u = x$, and rejecting the solution $x = 1$, we obtain the biquadratic

$$cx^4 + 2cx^3 + 2c'x + c' = 0,$$

whose roots are

$$\operatorname{cn}(4K/3),\ \operatorname{cn}(4iK'/3),\ \operatorname{cn}\{2(K+iK')/3\},\ \operatorname{cn}\{2(K-iK')/3\}.$$

Writing

$$4c^2x^4 + 8c^2x^3 + 8cc'x + 4cc' \equiv (2cx^2 + 2cx + 2\theta)^2 - (2Mx + N)^2,$$

we find that

$$2\theta^3 = cc'; \qquad N^2 = (4cc')^{\frac{2}{3}}\{1-(4cc')^{\frac{1}{3}}\};$$

$$\theta_1 = (\tfrac{1}{2}cc')^{\frac{1}{3}} = \tfrac{1}{2}c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{4iK'}{3} + \operatorname{cn}\frac{2(K+iK')}{3}\operatorname{cn}\frac{2(K-iK')}{3}\right\},$$

$$\theta_2 = \omega(\tfrac{1}{2}cc')^{\frac{1}{3}} = \tfrac{1}{2}c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{2(K-iK')}{3} + \operatorname{cn}\frac{4iK'}{3}\operatorname{cn}\frac{2(K+iK')}{3}\right\},$$

$$\theta_3 = \omega^2(\tfrac{1}{2}cc')^{\frac{1}{3}} = \tfrac{1}{2}c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{2(K+iK')}{3} + \operatorname{cn}\frac{4iK'}{3}\operatorname{cn}\frac{2(K-iK')}{3}\right\};$$

$$N_1 = \pm c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{4iK'}{3} - \operatorname{cn}\frac{2(K+iK')}{3}\operatorname{cn}\frac{2(K-iK')}{3}\right\},$$

$$N_2 = \pm c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{2(K-iK')}{3} - \operatorname{cn}\frac{4iK'}{3}\operatorname{cn}\frac{2(K+iK')}{3}\right\},$$

$$N_3 = \pm c\left\{\operatorname{cn}\frac{4K}{3}\operatorname{cn}\frac{2(K+iK')}{3} - \operatorname{cn}\frac{4iK'}{3}\operatorname{cn}\frac{2(K-iK')}{3}\right\};$$

and, with this notation,

$$\begin{aligned}\zeta = \frac{1}{6}\Bigg[&\left\{\frac{8(1-cc')^2}{9\theta_1^4} - \frac{4(1-cc')}{3\theta_1^2} - 1\right\}\frac{(1+\theta_2)(1+\theta_3)}{\theta_2^2\theta_3^2}N_2N_3 - \\ &-\left\{\frac{8(1-cc')^2}{9\theta_2^4} - \frac{4(1-cc')}{3\theta_2^2} - 1\right\}\frac{(1+\theta_3)(1+\theta_1)}{\theta_3^2\theta_1^2}N_3N_1 - \\ &-\left\{\frac{8(1-cc')^2}{9\theta_3^4} - \frac{4(1-cc')}{3\theta_3^2} - 1\right\}\frac{(1+\theta_1)(1+\theta_2)}{\theta_1^2\theta_2^2}N_1N_2\Bigg] + \\ &+ 64\frac{(1-cc')^3}{27c^2c'^2} - 6.\end{aligned}$$

In the foregoing ω is taken as $\exp(2i\pi/3)$, and the identification of the two expressions for each of θ_1, θ_2, θ_3 and the determination of

the ambiguous signs of N_1, N_2, N_3 are effected by the use of the approximate solutions of the biquadratic,

$$\operatorname{cn}\frac{4K}{3} = -\frac{1}{2}, \qquad \operatorname{cn}\frac{4iK'}{3} = -\left(\frac{2}{c}\right)^{\frac{1}{3}},$$

$$\operatorname{cn}\frac{2(K+iK')}{3} = -\omega\left(\frac{2}{c}\right)^{\frac{1}{3}}, \qquad \operatorname{cn}\frac{2(K-iK')}{3} = -\omega^2\left(\frac{2}{c}\right)^{\frac{1}{3}}$$

when c is small. The actual signs of N_1, N_2, N_3 are still ambiguous, but must be all plus or all minus.

The group of transformations under which ζ is automorphic is, of course, a sub-group of the modular group. The generating substitutions are any two of

$$\text{(i) } \tau' = \tau+3, \text{ (ii) } \tau' = \tau/(1-\tau); \text{ (iii) } \tau' = (\tau-3)/(\tau-2).$$

The transformations of the modular group not belonging to this sub-group permute the four roots of the biquadratic from which we obtained ζ as a function of z, and thereby give the representations of the corresponding transformations of the triangle we have considered.

5. The functions of the triangles of angles $\pi/2$, $\pi/6$, 0 and $\pi/6$, $\pi/6$, 0. The triangle of the preceding section can also be built up of two triangles of angles $\pi/2$, $\pi/6$, 0. In fact, a circle with centre E and radius ED bisects the angle EDC, and, of course, cuts EF at right angles. (See Fig. 5.)

Mapping the triangle of angles $\pi/2$, $\pi/6$, 0 on a Z half-plane, we find the relation

$$Z = \frac{\zeta^2}{(\zeta-2)^2}.$$

The group of transformations appropriate to this function is not a sub-group of the modular group. This is evident on account of the additional circles which have been introduced into the figure. Two generating transformations are (i) $\tau' = \tau+3$, (ii) $\tau' = -3/\tau$.

Again, we can build up a triangle GDC of angles $\pi/6$, $\pi/6$, 0 from two repetitions of this last triangle, and mapping such a triangle on a Z' half-plane, we have

$$Z' = (1-Z)^{\frac{1}{2}} = \frac{2i(\zeta-1)^{\frac{1}{2}}}{2-\zeta}.$$

I have not, so far, been able to exhibit this as a one-valued function of τ.

Generating substitutions of the group of this function, which is a sub-group of the group of the triangle last considered, but not of the modular group, are (i) $\tau' = \tau + 6$, (ii) $\tau' = 3(\tau - 1)/\tau$.

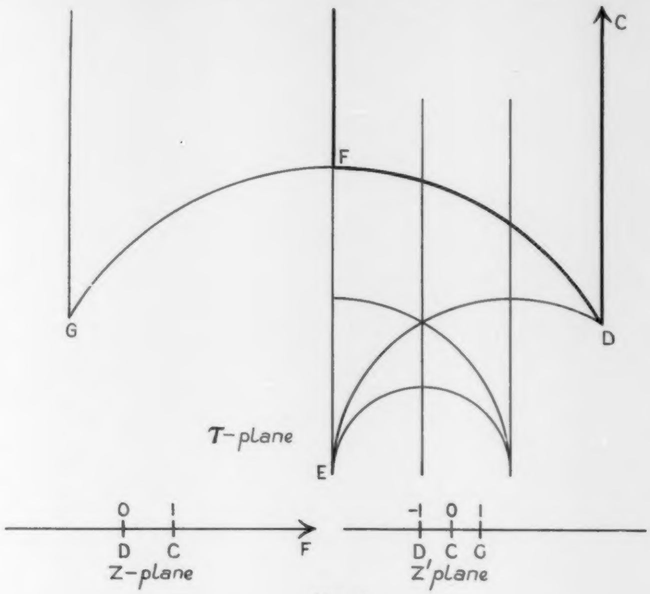


FIG. 5.

6. The functions of the triangles of angles $\pi/2$, $\pi/4$, 0 and $\pi/4$, $\pi/4$, 0. From two repetitions of a triangle ADC of angles $\pi/2$, $\pi/4$, 0, we can build up a triangle ABC of angles $\pi/2$, 0, 0 whose function has already been given. (See Fig. 6.)

We map the first on a Z half-plane, and the second on a ζ half-plane, and obtain the relation

$$Z = 4\zeta(1-\zeta) = \frac{16c^2c'}{(c-2)^4}.$$

Two transformations generating the group of this function are (i) $\tau' = \tau + 1$, (ii) $\tau' = -1/2\tau$. Again we note that the group is not included in the modular group.

From two repetitions of this last triangle we may build up a triangle ECA of angles $\pi/4$, $\pi/4$, 0.

We map this on a Z' half-plane, and find that

$$Z'^2 = \frac{Z}{Z-1} = \frac{-16c^2c'}{(c^2+4c-4)^2}.$$

Hence
$$Z' = \frac{4ik^2k'}{4-4k^2-k^4}.$$

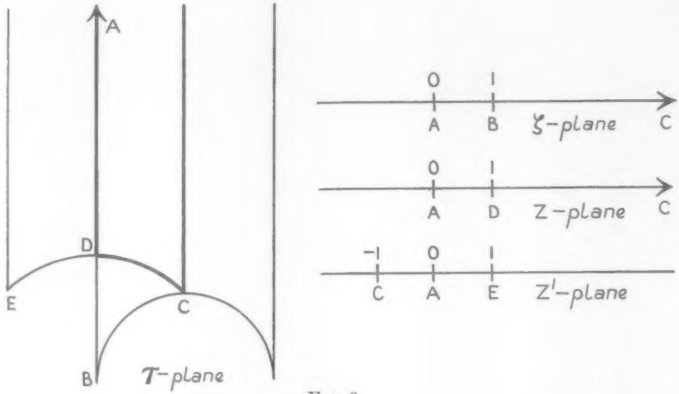


Fig. 6.

The group of this function is generated by

$$\text{(i) } \tau' = \tau+2, \quad \text{(ii) } \tau' = (2\tau-1)/2\tau.$$

It is a sub-group of the group of the preceding function, but not of the modular group.

[The triangle of three zero angles may be subdivided into triangles of angles $\pi/2$, $\pi/4$, 0 in three distinct ways. For minor considerations I have used different subdivisions in sections 2 and 6. Similar choices are possible in other places, so that most of the results can be given in various forms.]

LATTICE-POINT SUMMATION FORMULAE

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[Received 29 October 1930]

1. Introduction

1.1. Our main object in writing this paper has been to obtain a reasonably straightforward and self-contained proof of Voronoï's summation-formula* for $\sum d(n)f(n)$, where $d(n)$ denotes the number of divisors of n. Such a formula may be regarded as a formula for summation over the lattice points which lie between a rectangular hyperbola and its asymptotes.

The scheme of the proof may be illustrated by recalling Franel's† investigation of the Euler-Maclaurin sum formula. He started with the formulae

$$[x]-x+\tfrac{1}{2}=\sum_{m=1}^{\infty}\frac{\sin 2m\pi x}{m\pi}, \tag{A}$$

$$\sum_{r=0}^{h} F(r)=\tfrac{1}{2}F(0)+\tfrac{1}{2}F(h)+\int_0^h F(x)\,dx-\int_0^h\{[x]-x+\tfrac{1}{2}\}F'(x)\,dx, \tag{B}$$

taking h to be a positive integer.

In (B) he put the series-form of (A) and integrated by parts for each term, the result being

$$\sum_{r=0}^{h} F(r)=\tfrac{1}{2}F(0)+\tfrac{1}{2}F(h)+\int_0^h F(x)\,dx+2\sum_{m=1}^{\infty}\int_0^h \cos 2m\pi x\,F(x)\,dx. \tag{C}$$

It should be noticed that if

$$F(x)=\frac{\sin 2\pi xy}{\pi x}$$

and h becomes infinite, the formula (C) becomes

$$\sum_{r=0}^{\infty}\frac{\sin 2r\pi y}{r\pi}=y+\tfrac{1}{2}+[y], \tag{C, A}$$

which is merely another way of writing (A). Accordingly, the proof of (C) is obtained by first proving a particular case of it, namely, when

$$F(x)=\int_0^x 2\cos 2m\pi t\,dt=D^{-1}(2\cos 2m\pi x),$$

* Voronoï: *Annales de l'École Normale*, (3) 21 (1904), 207–67, 459–533; at p. 529.

† *Math. Annalen*, 47 (1896), 433–40; cf. Voronoï, loc. cit., p. 207.

substituting the result in the relatively trivial formula (B), and integrating each term by parts.

In the corresponding problem for lattice points in two dimensions, questions of convergence make it convenient* to start further back and to consider the particular case which corresponds to

$$D^{-2}(2\cos 2m\pi x).$$

Franel's equation (A) is a series-expansion of

$$[x]-x.$$

In the same way, we arrive at the functions appropriate to the sum $\sum d(n)f(n)$ by finding a series-expansion of

$$d(1)+d(2)+\ldots+d([x])-[\text{the dominant term of this sum}], \tag{D}$$

or rather, beginning farther back, by first finding a series for the integral of this, i.e. we investigate

$$\sum_{n<x}(x-n)d(n)-[\text{the dominant term of this sum}]. \tag{E}$$

The formula which corresponds to (B), namely (4.33), comes quite simply in the course of this investigation and the final formula, corresponding to (C), follows, in the manner of Franel's proof, on substituting a series in (4.33) and integrating by parts for each term. The series we obtain for (D) and (E) are, as one would expect, particular cases of the final formula.

1.2. In the investigation of the two sums

$$\sum_{n<x} d(n), \qquad \sum_{n<x}(x-n)d(n),$$

we have gone beyond the minimum of work necessary to the proof of Voronoï's formula. This was inevitable when we found that our work led naturally to several new and interesting facts.

We consider the expansion in an infinite series of

$$\frac{1}{\Gamma(\alpha)}\sum_{n<x}(x-n)^{\alpha-1}d(n), \tag{1.21}$$

where α is not necessarily an integer. In doing so we develop a new method† of establishing such series. In our particular example (1.21)

* It is not necessary to do so, but the saving in arithmetical detail is considerable.

† It has been pointed out to us by Professor Hardy that Landau used this method for a corresponding problem. His example is in one way more general, in another way less general than ours. As applied to our problem, where $d(n)$

the development of the work is governed by the fact that the arithmetical function $d(n)$ is linked up with the Riemann zeta function by means of the equation

$$\{\zeta(s)\}^2 = \sum d(n)n^{-s}, \qquad R(s) > 1, \tag{1.22}$$

and this function satisfies the simple functional equation

$$\{\zeta(s)\}^2 = \{\tfrac{1}{2}(2\pi)^s/\Gamma(s)\cos\tfrac{1}{2}s\pi\}^2\{\zeta(1-s)\}^2. \tag{1.23}$$

It will be seen, from the details of § 2, that our method will apply, at least in its early stages, to other arithmetical functions $\alpha(n)$ such that the series

$$\sum \alpha(n)n^{-s}$$

is a function $\psi(s)$ having a simple functional equation of the type

$$\psi(s) = A(s)\psi(1-s).$$

Actually, the first expression we considered was not (1.21) but

$$\frac{1}{\Gamma(\alpha)}\sum_{n<x}(x-n)^{\alpha-1}r(n), \tag{1.24}$$

where $r(n)$ is the number of ways of expressing n as the sum of two squares. The expansion of this function in the form*

$$\frac{1}{\Gamma(\alpha)}\sum_{n=1}^{n<x}(x-n)^{\alpha-1}r(n)$$
$$= \frac{\pi x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \pi^{1-\alpha}x^{\frac{1}{2}\alpha}\sum_{r=1}^{\infty} n^{-\frac{1}{2}\alpha}r(n)J_\alpha\big(2\pi\sqrt{(nx)}\big) \tag{1.25}$$

has received considerable attention and we obtained no new information about it, although our proofs appeared to be simpler than most of the known proofs. The arithmetic necessary to prove (1.25) is much simpler than that of the present investigation and all the facts required, e.g. asymptotic values, and the values of certain integrals, are readily available.

1.3. The expansion of (1.21) for integer values of α was established

replaces the more general $d_k(n)$, his work deals directly only with $\alpha = 3$ (cf. § 2). For details, see Landau, 'Über Dirichlets Teilerproblem', *Sitzungsberichte zu München* (1915), pp. 317–28.

Various improvements in detail have been adopted at the instance of Professor Hardy, whose helpful criticisms we gratefully acknowledge.

* Hardy, 'The average orders of the functions $P(x)$ and $\Delta(x)$': *Proc. London Math. Soc.* (2) 15 (1916), 205.

by Voronoï in 1904 and, since that time, various allied sums have been considered by other writers.* In these investigations the functions which replace the J_α of (1.25) are the Bessel functions $Y_\alpha \pm (2/\pi)K_\alpha$ of argument $4\pi\sqrt{(nx)}$.

We consider both integer and non-integer values of α, and discover the interesting fact that, for general values of α, the appropriate functions are not, in fact, Bessel functions, but functions which are identical with them only when α is an integer. These functions are most easily described as a sum of terms of the type†

$$\frac{(\frac{1}{2}z)^{2m}}{\Gamma(m+1)\Gamma(m+\alpha+1)}\{2\log(\tfrac{1}{2}z)-\psi(m+1)-\psi(m+\alpha+1)\}. \quad (1.31)$$

It is only when α is an integer that the infinite series in the expansions of

$$(\tfrac{1}{2}z)^{-\alpha}Y_\alpha(z), \qquad (\tfrac{1}{2}z)^{-\alpha}K_\alpha(z)$$

consist of terms of this type, but the form (1.31) persists as the typical term in the expansions of the functions we use, whether α is an integer or not.

From another point of view this was to be expected, since an application of Voronoï's formula to the sum

$$\frac{1}{\Gamma(\alpha)}\sum_{n<x}(x-n)^{\alpha-1}d(n)$$

shows that the functions we require are fractional integrals of

$$-Y_0\big(4\pi\sqrt{(nt)}\big)+\frac{2}{\pi}K_0\big(4\pi\sqrt{(nt)}\big),$$

and term-by-term integration will yield a series whose terms are of the type (1.31).

1.4. The sections 3 and 6 of the present paper outline enough of the theory of these functions to enable us to proceed with our major problem. These sections may be regarded as parentheses inserted in the main argument, and, in reading the other sections, only the definitions and results of sections 3 and 6 will be found necessary.

* e.g. Oppenheim, *Proc. London Math. Soc.* (2) 26 (1927), 295–350. This paper contains many references to the literature of the subject.

† $\psi(z) = \Gamma'(z)/\Gamma(z)$. Its properties are used freely throughout the paper.

2. The sums $\sum (x-n)^k d(n)$

2.1. We start from the formula, readily verified by the usual methods of contour integration,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\Gamma(s+\alpha)}\left(\frac{x}{n}\right)^s ds = \begin{cases} 0 & (0 \leqslant x \leqslant n) \\ \dfrac{1}{\Gamma(\alpha)}\left(1-\dfrac{n}{x}\right)^{\alpha-1} & (x > n) \end{cases} \tag{2.11}$$

provided that $c > 0$ and $\alpha > 1$.

2.2. It is well known that, writing $s = \sigma + it$,

$$\{\zeta(s)\}^2 = \sum_{n=1}^{\infty} d(n)n^{-s} \qquad (\sigma > 1), \tag{2.21}$$

$$d(n) = O(n^{\delta}) \qquad (\delta > 0). \tag{2.22}$$

We define the function $D_{\alpha-1}(x)$ by means of the equation

$$D_{\alpha-1}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\sum_{n \leqslant x}\left(1-\frac{n}{x}\right)^{\alpha-1} d(n). \tag{2.23}$$

Then, if $\alpha > 1$ and $\sigma > 0$, (2.11) shows that

$$D_{\alpha-1}(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{n \leqslant x}\frac{x^{s+\alpha-1}d(n)\Gamma(s)}{n^s\Gamma(s+\alpha)}\,ds. \tag{2.24}$$

If now $\sigma > 1$, the series $\sum d(n)n^{-\sigma}$ is convergent and it is not difficult to prove that

$$\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{n > x}\frac{x^{s+\alpha-1}d(n)\Gamma(s)}{n^s\Gamma(s+\alpha)}\,ds = 0. \tag{2.25}$$

Hence, on using (2.21), when $\alpha > 1$

$$D_{\alpha-1}(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\{\zeta(s)\}^2\frac{x^{s+\alpha-1}\Gamma(s)}{\Gamma(s+\alpha)}\,ds \qquad (\sigma > 1). \tag{2.26}$$

2.3. If $\alpha > 2$ and $0 < c < \mathrm{Min}(\frac{1}{2}, \frac{1}{2}\alpha - 1)$, we have, by using the

* The key to the situation, here and elsewhere in § 2, is the formula

$$|\Gamma(\sigma \pm it)| = e^{-\frac{1}{2}\pi t}t^{\sigma-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\{1+o(1)\}$$

where $o(1) \to 0$ as $t \to \infty$, uniformly for any finite range of σ.

The result (2.11) is not new. In a slightly modified form it is given by Hardy and Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tract, No. 18), p. 52.

obvious contour* and calculating the residues of the integrand at $s=0$ and $s=1$,

$$D_{\alpha-1}(x)=\frac{1}{2\pi i}\int\limits_{-c-i\infty}^{-c+i\infty}\{\zeta(s)\}^2\frac{x^{s+\alpha-1}\Gamma(s)}{\Gamma(s+\alpha)}\,ds+$$
$$+\frac{x^{\alpha-1}}{4\Gamma(\alpha)}+\frac{x^{\alpha}}{\Gamma(1+\alpha)}\{\gamma+\log x-\psi(1+\alpha)\}. \quad (2.31)$$

By means of the functional equation

$$\zeta(s)=\tfrac{1}{2}(2\pi)^s\zeta(1-s)/\Gamma(s)\cos(\tfrac{1}{2}s\pi)$$

and the expansion (2.21), we may write this last integral as

$$\frac{1}{8\pi i}\int\limits_{-c-i\infty}^{-c+i\infty}\frac{(2\pi)^{2s}x^{s+\alpha-1}}{\Gamma(s)\Gamma(s+\alpha)\cos^2(\frac{1}{2}s\pi)}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-s}}\,ds. \quad (2.32)$$

On changing the order of summation and integration,† this becomes

$$\frac{1}{4^{\alpha}\pi^{2\alpha-2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\alpha}}\frac{1}{2\pi i}\int\limits_{-c-i\infty}^{-c+i\infty}\frac{(4\pi^2nx)^{s+\alpha-1}}{\Gamma(s)\Gamma(s+\alpha)\cos^2(\frac{1}{2}s\pi)}\,ds. \quad (2.33)$$

2.4. When n and x are fixed, the integral in (2.33) is equal to minus $2\pi i$ times the sum of the residues of the integrand at its double poles $s=2m+1$ $(m=0,1,2,\ldots)$. The residue at $s=2m+1$ is‡

$$\frac{4}{\pi^2}\cdot\frac{(4\pi^2nx)^{2m+\alpha}}{\Gamma(2m+1)\Gamma(2m+1+\alpha)}\{\log(4\pi^2nx)-\psi(2m+1)-\psi(2m+1+\alpha)\}.$$

Accordingly, when $\alpha>2$, we may write

$$D_{\alpha-1}(x)-\frac{x^{\alpha-1}}{4\Gamma(\alpha)}-\frac{x^{\alpha}}{\Gamma(1+\alpha)}\{\gamma+\log x-\psi(1+\alpha)\}$$
$$=-4x^{\alpha}\sum_{n=1}^{\infty}d(n)\sum_{m=0}^{\infty}\frac{(4\pi^2nx)^{2m}}{\Gamma(2m+1)\Gamma(2m+1+\alpha)}\times$$
$$\times\{2\log\bigl(2\pi\surd(nx)\bigr)-\psi(2m+1)-\psi(2m+1+\alpha)\}, \quad (2.41)$$

* If $\alpha=2+\theta$, $c<\frac{1}{2}\theta$, $|\zeta(\sigma\pm it)|=O(t^{\frac{1}{2}+c}\log t)$ when $-c\leqslant\sigma\leqslant 0$, $O(t^{\frac{1}{2}}\log t)$ when $0\leqslant\sigma\leqslant 1$, and $O(\log t)$ when $\sigma\geqslant 1$. Throughout, the integrand in (2.31) is $O(t^{-1-\delta})$, where $\delta>0$.

† On writing $\cos(\frac{1}{2}s\pi)=\pi/\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s)$ and using the asymptotic formula for the gamma function, it is readily seen that (2.32) is absolutely convergent. For large t, the dominant term is $t^{1+2c-\alpha}$.

‡ The residue at the double pole presents no difficulty if it is remembered that the coefficient of h in $f(x+h)/\phi(x+h)$ is

$$\frac{f}{\phi}\left(\frac{f'}{f}-\frac{\phi'}{\phi}\right)$$

evaluated at x.

or, by anticipating our later notation for the 'modified Bessel function',

$$= 2\pi x^{\alpha} \sum_{n=1}^{\infty} d(n)\lambda_{\alpha}\big(4\pi\sqrt{(nx)}\big). \tag{2.42}$$

2.5. *The particular case when α is a positive integer.* When α is a positive integer, a say, the term in (2.42) which involves $d(n)$ may be written

$$d(n)\frac{(x/n)^{\frac{1}{2}a}}{(2\pi)^{a-1}}\Big[-Y_a\big(4\pi\sqrt{(nx)}\big)+\frac{2}{\pi}\cos a\pi\, K_a\big(4\pi\sqrt{(nx)}\big)-$$

$$-\frac{1}{\pi}\sum_{m=0}^{a-1}\frac{(a-m-1)!}{m!}\{1+(-1)^{m+a}\}\big(2\pi\sqrt{(nx)}\big)^{2m-a}\Big]; \tag{2.51}$$

that is to say, *the coefficient of $d(n)$ contains the Bessel function*

$$-Y_a\big(4\pi\sqrt{(nx)}\big)\pm\frac{2}{\pi}K_a\big(4\pi\sqrt{(nx)}\big)$$

deprived of its initial terms, i.e. such terms in its expansion as do not involve a logarithm or a logarithmic derivate of a gamma function.

On the other hand, the sum of the terms in the second line of (2.51), for $n = 1, 2,\ldots$, can be dealt with in such a way that it appears only as a sum of zeta functions. Keeping m fixed and summing from $n = 1$ to $n = \infty$, we get

$$-\frac{x^m}{\pi}\cdot\frac{(a-m-1)!}{m!}\{1+(-1)^{m+a}\}(2\pi)^{2m-2a+1}\sum_{n=1}^{\infty}\frac{d(n)}{n^{a-m}},$$

which is a multiple of $\{\zeta(a-m)\}^2$.

Accordingly, we may write (2.41) in the form

$$D_{a-1}(x)-\frac{x^{a-1}}{4\Gamma(a)}-\frac{x^a}{\Gamma(1+a)}\{\gamma+\log x-\psi(1+a)\}$$

$$=\sum_{n=1}^{\infty}d(n)\frac{(x/n)^{\frac{1}{2}a}}{(2\pi)^{a-1}}\Big[-Y_a\big(4\pi\sqrt{(nx)}\big)+\frac{2}{\pi}\cos a\pi\, K_a\big(4\pi\sqrt{(nx)}\big)\Big]-$$

$$-\frac{1}{\pi}\sum_{m=0}^{a-2}\frac{(2\pi)^{2m-2a+1}x^m(a-m-1)!}{m!}\{1+(-1)^{m+a}\}\{\zeta(a-m)\}^2. \tag{2.52}$$

If we first put $a-m-1=\lambda$, note that alternate terms vanish, and then replace $\zeta(1+\lambda)$ by the proper multiple of $\zeta(-\lambda)$, these last terms may be written as

$$-\sum_{m=0}^{\frac{1}{2}[a]-1}\frac{x^{a-2m-2}\{\zeta(-2m-1)\}^2}{(2m+1)!(a-2m-2)!}. \tag{2.53}$$

Finally, denoting by $\Phi_{a-1}(x)$ the sum of the residues, *qua* function of s, of

$$\frac{\{\zeta(s)\}^2\Gamma(s)}{\Gamma(s+a)}x^{s+a-1}, \tag{2.54}$$

we may write (2.52) in the convenient form*

$$\frac{1}{\Gamma(a)}\sum_{n<x}(x-n)^{a-1}d(n)-\Phi_{a-1}(x)$$
$$=\sum_{n=1}^{\infty}d(n)\frac{(x/n)^{\frac{1}{2}a}}{(2\pi)^{a-1}}\Big[-Y_a\big(4\pi\sqrt{(nx)}\big)+\frac{2}{\pi}\cos a\pi\,K_a\big(4\pi\sqrt{(nx)}\big)\Big]. \tag{2.55}$$

2.6. *The case when $\alpha \leqslant 2$.*

So far, we have proved the identity (2.41), or (2.42), only when $\alpha > 2$. This limitation on α has been made to ensure absolute convergence at all points of the work, and so to ensure a simple justification of the limiting processes used. Before we can consider the case $\alpha \leqslant 2$, we must investigate the properties of $\lambda_\alpha(z)$; this, though not a Bessel function, has many of its characteristics.

3. Modified Bessel functions

3.1. Introducing the notation $Y_{/\nu}(z)$, let us consider the functions

$$Y_{/\nu}(z)=$$
$$\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-)^m(\frac{1}{2}z)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}\{2\log(\tfrac{1}{2}z)-\psi(m+1)-\psi(m+\nu+1)\}, \tag{3.11}$$

$$K_{/\nu}(z)=$$
$$\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}\{2\log(\tfrac{1}{2}z)-\psi(m+1)-\psi(m+\nu+1)\}. \tag{3.12}$$

They are derived in a fairly obvious way from the expansions of the Bessel functions, *n being now a positive integer*,

$$Y_n(z), \qquad (2/\pi)K_n(z).$$

Two details may be noted:† the division by $(\frac{1}{2}z)^n$ in both cases and, in the case of $K_n(z)$, the omission of any factor to represent $(-1)^{n+1}$.

* Voronoï, *Annales de l'École Normale*, (3) 21 (1904), gives his result in much the same form; cf. the last lines of pp. 485, 497 respectively. Oppenheim, *Proc. London Math. Soc.*, (2) 26 (1927), 336, 337, has a corresponding result which is connected with $\zeta(s)\zeta(s-k)$ in the way our (2.55) is connected with $\{\zeta(s)\}^2$. At equation (8.52) he has a trivial misprint, ρ for $\frac{1}{2}\rho$, in the last line.

† We have preferred to keep $Y_n(z)$ rather than $Y_n(2\sqrt{z})$ as our pattern, though the latter offers obvious simplifications. Its adoption would, however, obscure the relation between the two functions Y_n and $Y_{/n}$.

For convenience, we also use the notations

$$J_{/\nu}(z) = (\tfrac{1}{2}z)^{-\nu} J_\nu(z), \qquad I_{/\nu}(z) = (\tfrac{1}{2}z)^{-\nu} I_\nu(z). \tag{3.13}$$

Direct substitution in (3.11) and (3.12) will show that

$$Y_{/\nu}(zi) = K_{/\nu}(z) + iI_{/\nu}(z), \qquad K_{/\nu}(zi) = Y_{/\nu}(z) + iJ_{/\nu}(z), \tag{3.14}$$

$$K_{/\nu}(z) = -iJ_{/\nu}(zi) + Y_{/\nu}(zi) = iJ_{/\nu}(-zi) + Y_{/\nu}(-zi), \tag{3.15}$$

$$Y_{/\nu}(z) = -iI_{/\nu}(zi) + K_{/\nu}(zi) = iI_{/\nu}(-zi) + K_{/\nu}(-zi). \tag{3.16}$$

If two further functions $\lambda_\nu(z)$, $\mu_\nu(z)$ are defined by the equations

$$\lambda_\nu(z) = -Y_{/\nu}(z) - K_{/\nu}(z), \qquad \mu_\nu(z) = -Y_{/\nu}(z) + K_{/\nu}(z), \tag{3.17}$$

we get relations of the type

$$\lambda_\nu(zi) + iJ_{/\nu}(zi) = \lambda_\nu(z) - iJ_{/\nu}(z), \tag{3.18}$$

a result which is extremely useful in the calculation of certain integrals required later on in the main argument.

3.2. It is useful to notice that, when $R(\nu) > 0$,

$$\frac{2}{\pi}\lambda_\nu(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(\frac{1}{2}z)^{2s-2}\sec^2(\frac{1}{2}s\pi)}{\Gamma(s)\Gamma(s+\nu)}\,ds, \tag{3.21}$$

where c is a number, lying between -1 and $+1$, such that

$$R(\nu) + 2c - 2 > 0.$$

Further, the expansion of $\lambda_\nu(z)$ is given by

$$\lambda_\nu(z) = -\frac{2}{\pi}\sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{4m}}{\Gamma(2m+1)\Gamma(2m+\nu+1)}\{2\log(\tfrac{1}{2}z) - \psi(2m+1) - \psi(2m+\nu+1)\}. \tag{3.22}$$

For comparison with the L, M functions introduced by Hardy, we note also that

$$\left.\begin{aligned} \lambda_0(z) &= -Y_0(z) + (2/\pi)K_0(z) = M_0(z), \\ \mu_0(z) &= -Y_0(z) - (2/\pi)K_0(z) = L_0(z), \\ \tfrac{1}{2}z\lambda_1(z) &= -Y_1(z) - (2/\pi)K_1(z) = L_1(z). \end{aligned}\right\} \tag{3.23}$$

3.3. *Asymptotic expansions.*

Suppose now that ν is not a whole number and consider the integral*

$$-\frac{1}{2\pi i}\int_{-\infty i-p+\frac{1}{2}}^{\infty i-p+\frac{1}{2}} \frac{\Gamma(1-s)\Gamma(1-\nu-s)(\frac{1}{2}z)^{2s-2}}{\sin s\pi}\,ds, \tag{3.31}$$

* The integral is practically that obtained by putting $\mu = \nu - 1$ in p. 351 of Watson's *Theory of Bessel Functions*. Our function $Y_{/\nu}(z)$ is expressible, as W. p. 349 (3) shows, in terms of a Lommel function $S_{\nu-1,\nu}(z)$ and $Y_\nu(z)$.

where the integer p is large enough to ensure that the point $s = 1-\nu$ lies to the right of the path of integration.

If $|\arg z| \leqslant \pi-\delta$, it can be shown, by the help of Stirling's formula, that (3.31) is the sum of the residues of the integrand at those poles which lie to the right of the path of integration. These poles fall into three groups:

(i) double poles at $s = 1, 2, 3,\ldots,$

(ii) simple poles at $s = 1-\nu, 2-\nu,\ldots,$

(iii) simple poles at $s = 0, -1, -2,\ldots, -p+1,$

and in dealing with (i) it is easier to write the integrand as

$$\frac{\pi^2(\frac{1}{2}z)^{2s-2}}{\Gamma(s)\Gamma(\nu+s)\sin(\nu+s)\pi\sin^2 s\pi}.$$

The residues at these poles are

(i) $s = r$, $\qquad \dfrac{2\log(\frac{1}{2}z)-\psi(r)-\psi(\nu+r)-\pi\cot\nu\pi}{\Gamma(r)\Gamma(\nu+r)\sin(\nu+r)\pi}(\frac{1}{2}z)^{2r-2};$

(ii) $s = r-\nu$, $\qquad (-)^r\pi(\frac{1}{2}z)^{2r-2\nu-2}/\Gamma(r)\Gamma(r-\nu)\sin^2\nu\pi;$

(iii) $s = -r$, $\qquad (-)^r\Gamma(1+r)\Gamma(1-\nu+r)(\frac{1}{2}z)^{-2r-2}/\pi.$

Accordingly, (3.31) is equal to

$$\frac{\pi}{\sin\nu\pi}\{-Y_{/\nu}(z)+(\tfrac{1}{2}z)^{-\nu}Y_\nu(z)\}+\frac{1}{\pi}\sum_{r=0}^{p-1}\frac{\Gamma(1+r)\Gamma(1-\nu+r)}{(-)^r(\frac{1}{2}z)^{2r+2}}. \tag{3.32}$$

On the other hand, when $|\arg z| \leqslant \pi-\delta$, the substitution $S = s+p$ in (3.31) shows it to be $O(\frac{1}{2}z)^{-2p-2}$. Hence, for large values of z,

$$Y_{/\nu}(z)-(\tfrac{1}{2}z)^{-\nu}Y_\nu(z) \sim \frac{1}{\pi}\sum_{r=0}^{\infty}\frac{\Gamma(1+r)}{\Gamma(\nu-r)}(\tfrac{1}{2}z)^{-2r-2}, \tag{3.33}$$

it being understood, for the moment, that ν is not an integer.

3.31. *The asymptotic expansion for integer values of ν.*

When ν is a positive integer, n say, the right-hand side of (3.33) becomes, on writing $m = n-1-r$,

$$\frac{1}{\pi}(\tfrac{1}{2}z)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}(\tfrac{1}{2}z)^{-n+2m}$$

and we see that the asymptotic relation (3.33) is now an equation. This equation is found to be a mere rearrangement of the expansion of $(\frac{1}{2}z)^{-n}Y_n(z)$.

When ν is a negative integer, $-n$ say, the asymptotic formula is no longer true. It must be replaced by

$$Y_{/-n}(z)-(\tfrac{1}{2}z)^{n}Y_{-n}(z)=\frac{(-)^{n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}(\tfrac{1}{2}z)^{2m}.$$

3.4. *Asymptotic expansions for $K_{/\nu}(z)$.*

From (3.15) we see that

$$K_{/\nu}(z)-i(-\tfrac{1}{2}zi)^{-\nu}H_{\nu}^{(2)}(-zi)=Y_{/\nu}(-zi)-(-\tfrac{1}{2}zi)^{-\nu}Y_{\nu}(-zi). \quad (3.41)$$

Transform the left-hand side of this by means of the known formulae*

$$H_{\nu}^{(2)}(xe^{-\frac{1}{2}\pi i})=-e^{\nu\pi i}H_{\nu}^{(1)}(xe^{\frac{1}{2}\pi i}),$$

$$K_{\nu}(x)=\tfrac{1}{2}\pi ie^{\frac{1}{2}\nu\pi i}H_{\nu}^{(1)}(xe^{\frac{1}{2}\pi i}).$$

Then, provided that

$$|\arg z-\tfrac{1}{2}\pi|<\pi, \text{ i.e. } -\tfrac{1}{2}\pi<\arg z<\tfrac{3}{2}\pi,$$

we have

$$K_{/\nu}(z)+\frac{2}{\pi}e^{\nu\pi i}(\tfrac{1}{2}z)^{-\nu}K_{\nu}(z)\sim\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{\Gamma(1+r)}{\Gamma(\nu-r)}\left(\frac{2i}{z}\right)^{2r+2}. \quad (3.42)$$

Here again, when ν is a positive integer, n say, the asymptotic formula reduces to an identity which is merely a rearrangement of the expansion of $(2/\pi)K_{n}(z)$.

If $\arg z$ is such that

$$|\arg z+\tfrac{1}{2}\pi|<\pi, \text{ i.e. } -\tfrac{3}{2}\pi<\arg z<\tfrac{1}{2}\pi,$$

the left-hand side of (3.42) must be replaced by†

$$K_{/\nu}(z)+\frac{2}{\pi}e^{-\nu\pi i}(\tfrac{1}{2}z)^{-\nu}K_{\nu}(z). \quad (3.43)$$

The two asymptotic relations have a common range of validity, namely, $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$, and the formula

$$K_{\nu}(z)\sim(\pi/2z)^{\frac{1}{2}}e^{-z}$$

shows that the two relations are consistent.

3.5. Finally, taking the case $-\frac{1}{2}\pi<\arg z<\frac{3}{2}\pi$,

$$\lambda_{\nu}(z)\sim(\tfrac{1}{2}z)^{-\nu}\left\{-Y_{\nu}(z)+\frac{2}{\pi}e^{\nu\pi i}K_{\nu}(z)\right\}-\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{\Gamma(2r)}{\Gamma(\nu-2r+1)}\left(\frac{2}{z}\right)^{4r}, \quad (3.51)$$

$$\mu_{\nu}(z)\sim(\tfrac{1}{2}z)^{-\nu}\left\{-Y_{\nu}(z)-\frac{2}{\pi}e^{\nu\pi i}K_{\nu}(z)\right\}-\frac{2}{\pi}\sum_{r=0}^{\infty}\frac{\Gamma(2\nu+1)}{\Gamma(\nu-2r)}\left(\frac{2}{z}\right)^{4r+2}, \quad (3.52)$$

* Watson, loc. cit., p. 75 (6) and p. 78 (8) respectively.

† We now use the first two members of (3.15) instead of the first and last.

and each expansion reduces to a trivial identity if ν is a positive integer.

3.6. Returning now to the formulae of § 2, we see that (2.42), by reason of the definitions (3.11), (3.12), and (3.17), is an alternative form of (2.41).

4. Convergence of the expansions

4.1. We have, in § 2, proved that, when $\alpha > 2$,

$$\frac{1}{\Gamma(\alpha)}\sum_{n<x}(x-n)^{\alpha-1}d(n)-\frac{x^{\alpha-1}}{4\Gamma(\alpha)}-\frac{x^{\alpha}}{\Gamma(1+\alpha)}\{\gamma+\log x-\psi(1+\alpha)\}$$

$$=2\pi x^{\alpha}\sum_{n=1}^{\infty}d(n)\lambda_{\alpha}\big(4\pi\surd(nx)\big). \quad (4.11)$$

Having now, in § 3, obtained an asymptotic formula for $\lambda_\alpha(z)$, we can at once deduce the validity of the above result when $\alpha > \frac{3}{2}$. It should be noted that the case $\alpha = 2$ is a vital one in our subsequent argument (§ 4.4, § 5), which starts from the assumption that (4.11) is true when $\alpha = 2$.

From (3.51) and the known asymptotic formulae for $Y_\nu(z)$ and $K_\nu(z)$, we have, when nx is large,

$$\lambda_{\alpha}\big(4\pi\surd(nx)\big)=(nx)^{-\frac{1}{2}\alpha-\frac{1}{4}}\big[A\sin\big(4\pi\surd(nx)-\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\pi\big)+Be^{-4\pi\surd(n\pi)}\big]+$$
$$+O\{(nx)^{-\frac{1}{2}\alpha-\frac{3}{4}}\}+O\{(nx)^{-2}\}. \quad (4.12)$$

Here A, B are bounded for all values of α, n, x and the constant implied in the order-symbol O is independent of α over any finite part of the α-plane. Hence, if $\alpha > \frac{3}{2}$, the series (4.11) converges absolutely, uniformly with regard to x in any closed x-interval which excludes the origin. Moreover, for a fixed x, the convergence is uniform with regard to α in any finite part of a strip $R(\alpha) \geqslant \frac{3}{2}+\delta$, both sides of (4.11) are analytic functions of α in such a strip, and so the equality of the two sides is proved whenever $R(\alpha) > \frac{3}{2}$.

Or again, since differentiating the terms of (4.11) with regard to x merely changes α into $\alpha-1$, term-by-term differentiation gives a uniformly convergent series if $\alpha-1 > \frac{3}{2}$, and accordingly (4.11) is valid when $\alpha > \frac{3}{2}$.

4.2. When we go on to values of $\alpha \leqslant \frac{3}{2}$, the series cannot be absolutely convergent and the whole problem of convergence* becomes

* An easy way of obtaining most of the facts is to quote the theorem,

more difficult. Our method of approach is to use a summation formula which is, essentially, the analogue of the formula (B) given in the introduction.

4.3. *A summation formula.*

As in § 2, let

$$D_0(x) = \sum_{n \leqslant x} d(n) = d(1)+d(2)+\ldots+d([x]),$$

so that $D_0(x)$ is zero if $x < 1$; further, let

$$r_0(x) = D_0(x) - x\{\gamma + \log x - \psi(2)\} - \tfrac{1}{4}, \tag{4.31}$$

$$r_1(x) = \int_0^x r_0(t)\,dt = \sum_{n \leqslant x} (x-n)d(n) - \tfrac{1}{2}x^2\{\gamma+\log x - \psi(3)\} - \tfrac{1}{4}x. \tag{4.32}$$

Using Stieltjes integrals and assuming a, b to be positive numbers, we have

$$\sum_{[a+1]}^{[b]} f(n)d(n) = \int_a^b f(t)\,dD_0(t) = \int_a^b f(t)\,dr_0(t) + \int_a^b (2\gamma+\log t)f(t)\,dt,$$

or, on integrating by parts,

$$\Big[f(t)r_0(t)\Big]_a^b - \int_a^b f'(t)r_0(t)\,dt + \int_a^b (2\gamma+\log t)f(t)\,dt.$$

Another integration by parts gives the formula

$$\sum_{[a+1]}^{[b]} f(n)d(n) = \Big[r_0(t)f(t) - r_1(t)f'(t)\Big]_a^b + $$

$$+ \int_a^b r_1(t)f''(t)\,dt + \int_a^b (2\gamma+\log t)f(t)\,dt. \tag{4.33}$$

In the applications of this formula we shall assume the elementary result* that, for large values of x,

$$r_0(x) = O(x^{\frac{1}{2}}).$$

given by Hardy, *Proc. London Math. Soc.* (2) 15 (1916), 1–25, at pp. 18, 19,

$$\sum d(n)n^{-\lambda}e^{-4\pi i\sqrt{(nx)}}$$

converges (i) *for* $\lambda > \frac{1}{2}$ *when x is not an integer, the convergence being uniform in any x-interval free from integers;*

(ii) *for* $\lambda > \frac{3}{4}$ *when x is an integer.*

The proof assumes considerable knowledge of difficult theorems about functions represented by Dirichlet series.

The theorem fails to deal directly with the critical case '$\alpha = 1$, x an integer', which requires a separate discussion; cf. Hardy, 'On the expression of a number as the sum of two squares': *Quart. J. of Math.* 46 (1915), 277, where the corresponding $r(n)$ problem is completely solved.

* Dirichlet, *Werke*, vol. ii, 49–66.

It is an immediate consequence of (4.11) with $\alpha = 2$, that

$$r_1(x) = O(x^{\frac{3}{4}}).$$

Now put, with $x > 0$, $\quad f(t) = \lambda_\alpha\big(4\pi\sqrt{(xt)}\big). \qquad (4.34)$

We find that the right-hand side of (4.33) reduces, for large values of a, to the form*

$$O(a^{\frac{1}{4}-\frac{1}{2}\alpha}) + \int_a^b r_1(t) f''(t)\,dt + O(a^{\frac{1}{4}-\frac{1}{2}\alpha}\log a), \tag{4.35}$$

when x lies between fixed positive numbers δ, K.

4.4. *The integral in* (4.35).

We know from 4.1 that

$$r_1(t) = 2\pi t^2 \sum_{n=1}^{\infty} d(n)\lambda_2\big(4\pi\sqrt{(nt)}\big).$$

The series on the right, whose terms are comparable with $n^\delta/(nt)^{1+\frac{1}{4}}$ for large values of n, is absolutely convergent, uniformly with regard to t in (a, b). Also, $f''(t)$ is bounded and integrable in (a, b). Hence

$$\int_a^b r_1(t) f''(t)\,dt = 2\pi \sum_{n=1}^{\infty} d(n) \int_a^b t^2 \lambda_2\big(4\pi\sqrt{(nt)}\big) f''(t)\,dt. \tag{4.41}$$

A little calculation shows that

$$f''(t) = \frac{2}{\pi\Gamma(\alpha+1)}\cdot\frac{1}{t^2} + 16\pi^4 x^2 \lambda_{\alpha+2}\big(4\pi\sqrt{(xt)}\big), \tag{4.42}$$

and, on using (3.51), we see that the term

$$\frac{2}{\pi\Gamma(\alpha+1)}\cdot\frac{1}{t^2}$$

cancels against the corresponding term of the asymptotic expansion of the function $\lambda_{\alpha+2}$. Accordingly,

$$f''(t) \sim A x^{\frac{3}{4}-\frac{1}{2}\alpha} t^{-\frac{1}{2}\alpha-\frac{5}{4}} \sin\{4\pi\sqrt{(xt)} - \tfrac{1}{2}\alpha\pi - \tfrac{5}{4}\pi\} + O(x^{-2}t^{-4}), \tag{4.43}$$

and so†

$$\int_a^b t^2 \lambda_2\big(4\pi\sqrt{(nt)}\big) f''(t)\,dt$$

$$\sim B n^{-\frac{5}{4}} x^{\frac{3}{4}-\frac{1}{2}\alpha} \int_a^b t^{-\frac{1}{2}\alpha-\frac{1}{2}} \cos\{4(\sqrt{n}\pm\sqrt{x})\sqrt{t} - \tfrac{5}{4} \mp (\tfrac{5}{4}+\tfrac{1}{2}\alpha)\}\pi\,dt + $$
$$+ O(n^{-\frac{5}{4}} x^{-2} a^{-\frac{9}{4}}). \tag{4.44}$$

* We find that

$$f'(t) = -x\left\{4\pi^2\mu_{\alpha+1}\big(4\pi\sqrt{(xt)}\big) + \frac{2}{\pi x t\,\Gamma(\alpha+1)}\right\}.$$

Further, the term in $(1/t)$ cancels against the corresponding term in the asymptotic expansion of the μ-function.

† A, B are absolute constants. The alternative signs in (4.44) indicate the presence of two terms, one with the upper signs and one with the lower.

4.5. *Convergence of the series* (4.11).

It is evident that when x is an integer, m say, a term arises in (4.44) when $n = m$, namely,

$$m^{-\frac{1}{4}} \int_a^b t^{-\frac{1}{2}\alpha-\frac{1}{4}} \cos \tfrac{1}{2}\alpha\pi \, dt,$$

which is quite unlike the other terms. On the other hand, this term is zero when $\alpha = 1$.

From (4.35) we see that $\alpha > \frac{1}{2}$ is a first condition for proving convergence, and we see from (4.41) and (4.44), on evaluating the order of the terms in the latter, that

(i) *When x is an integer, the series*

$$2\pi x^{\alpha} \sum d(n)\lambda_{\alpha}\big(4\pi\sqrt{(nx)}\big)$$

converges for $\alpha \geqslant 1$;

(ii) *When x is not an integer, the series converges for $\alpha > \frac{1}{2}$, the convergence being uniform with regard to x in any closed interval free from integer values*;

(iii) *If $\alpha > 1$, the series converges uniformly* with regard to x in any finite interval $0 < \delta \leqslant x \leqslant K$.*

Moreover, when x is fixed and is not an integer, the convergence of the series is uniform with regard to α in any finite part of a strip $R(\alpha) \geqslant \frac{1}{2}+\delta$, and, when x is a fixed integer, the convergence is uniform with regard to α in any finite part of a strip $R(\alpha) \geqslant 1+\delta$. Accordingly, by analytic continuation, *the identity* (4.11) *is true when $\alpha > 1$ for all positive values of x and when $\alpha > \frac{1}{2}$ for all positive non-integer values of x.*

Here again, as in 4.1, having established the uniform convergence of our series, we may use differentiation to prove (4.11) true either when $\alpha > 1$ and $x > 0$, or when $\alpha > \frac{1}{2}$ and x is positive but not an integer.

In the critical case when $\alpha = 1$ and x is a positive integer, we have proved the convergence of the infinite series in (4.11), but we have not proved the (known) result†

$$\sum_{n \leqslant m} d(n) - \tfrac{1}{2}d(m) - \tfrac{1}{4} - m(\log m + 2\gamma - 1) = 2\pi m \sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(nm)}\big). \tag{4.51}$$

* Cf. Oppenheim, loc. cit., p. 299, where a similar result is arrived at for the case corresponding to $\alpha = 2$.

† Various proofs, most of them difficult, are available; e.g. Oppenheim, loc. cit., pp. 344–50.

4.6. In the next section we return to the summation formula (4.33) and use it to prove (4.51). At the end of section 5 we indicate how to prove (4.11), for its full range of values of α and x, and (4.51) simultaneously, assuming only that our identity (4.11) is true for $\alpha > 2$.

There is, then, at this point of the argument, a choice of procedure. We have chosen the process with the lighter burden of arithmetical detail.

5. The sum $\sum d(n)$

5.1. In this section we use the summation formula (4.33) to prove the one case, namely, '$\alpha = 1$ and x an integer', which has, so far, escaped our analysis. The major interest of the proof is the way in which the discontinuous integrals of the λ-functions furnish the sum of the infinite series

$$\sum d(n)\lambda_1\big(4\pi\sqrt{(mn)}\big) \tag{5.11}$$

as a finite number of terms.

It may be noted, also, that the chief difficulty of the problem arises from the fact that whilst zero is a lower limit of integration for our discontinuous integrals, the function $\lambda_1\big(4\pi\sqrt{(mt)}\big)$ has a logarithmic singularity at $t = 0$. This difficulty we overcome by evaluating the difference between (5.11) and a series whose sum we already know, cf. (5.21) below.

When the method is applied to the series

$$m^{\frac{1}{2}} \sum \frac{r(n)}{n^{\frac{1}{2}}} J_1\big(2\pi\sqrt{(nm)}\big),$$

which is associated with lattice points of a circle, this difficulty does not arise. The origin does not then, as now, present a logarithmic singularity and we can evaluate the series directly.

5.2. In the summation formula (4.33) we make the substitution

$$f(t) = \lambda_1\big(4\pi\sqrt{(xt)}\big) - \lambda_1\big(4\pi\sqrt{(mt)}\big), \tag{5.21}$$

where m is a positive integer and $m-1 < x < m$. As before, we may write

$$r_1(t) = 2\pi t^2 \sum_{n=1}^{\infty} d(n)\lambda_2\big(4\pi\sqrt{(nt)}\big). \tag{5.22}$$

Moreover, it is easily seen, by using the asymptotic expansions of $\lambda_2\big(4\pi\sqrt{(nt)}\big)$ and $\lambda_3\big(4\pi\sqrt{(xt)}\big)$, that we may put $b = \infty$.

Accordingly, if δ is any positive number less than unity,

$$\sum_{n=1}^{\infty} f(n)d(n) = -r_0(\delta)f(\delta)+r_1(\delta)f'(\delta) + \int_{\delta}^{\infty} (2\gamma+\log t)f(t)\,dt +$$

$$+32\pi^5 \sum_{n=1}^{\infty} d(n) \int_{\delta}^{\infty} t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)\big\{x^2\lambda_3\big(4\pi\sqrt{(xt)}\big)-m^2\lambda_3\big(4\pi\sqrt{(mt)}\big)\big\}\,dt, \quad (5.23)$$

where the value of $f''(t)$ is obtained from (4.42).

5.3. Now for small values of δ, we have

$$r_0(\delta) = -\tfrac{1}{4}+O(\delta\log\delta), \qquad r_1(\delta) = -\tfrac{1}{4}\delta+O(\delta^2\log\delta),$$

$$f(\delta) = -\frac{2}{\pi}\log\frac{x}{m}+O(\delta^2\log\delta), \qquad f'(\delta) = O(\delta\log\delta).$$

It follows that*

$$-r_0(\delta)f(\delta)+r_1(\delta)f'(\delta) = -\frac{1}{2\pi}\log\frac{x}{m}+O(\delta\log\delta). \quad (5.31)$$

5.4. We have proved, in 4.5, that the series (5.22) is uniformly convergent in $0<\delta\leqslant t\leqslant K$, but the series is not uniformly convergent right down to $\delta=0$. Accordingly we cannot, without further argument, put $\delta=0$ in (5.23).

Let us consider then

$$\sum_{n=1}^{\infty} d(n) \int_{0}^{\delta} t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)\big\{x^2\lambda_3\big(4\pi\sqrt{(xt)}\big)-m^2\lambda_3\big(4\pi\sqrt{(mt)}\big)\big\}\,dt. \quad (5.41)$$

The coefficient of $d(n)$ may be written as

$$\frac{2}{(16\pi^2 n)^3} \int_{0}^{4\pi\sqrt{(n\delta)}} \theta^5\lambda_2(\theta)\left\{x^2\lambda_3\left(\theta\sqrt{\frac{x}{n}}\right)-m^2\lambda_3\left(\theta\sqrt{\frac{m}{n}}\right)\right\}\,d\theta. \quad (5.42)$$

Now, from the expansions of the λ-functions, we see that, for $0<\theta\leqslant 4\pi\sqrt{(n\delta)}$,

$$x^2\lambda_3\left(\theta\sqrt{\frac{x}{n}}\right)-m^2\lambda_3\left(\theta\sqrt{\frac{m}{n}}\right) = A+B\log(\theta n^{-\frac{1}{2}}), \quad (5.43)$$

where A, B are bounded for all values of n. Moreover, if δ is sufficiently small,

$$\log(\theta n^{-\frac{1}{2}})<0 \text{ when } 0<\theta\leqslant 4\pi\sqrt{(n\delta)},$$

so that (5.43) is less than a constant multiple of $\{1-\log(\theta n^{-\frac{1}{2}})\}$. Again, $\theta^{\frac{3}{2}}\lambda_2(\theta)$ tends to zero with θ and remains bounded as θ tends

* If we had taken $f(t)=\lambda_1(4\pi\sqrt{(mt)})$, we should have obtained a term $\frac{1}{2\pi}\log(m\delta)$, and this becomes infinite at $\delta=0$.

to infinity. Accordingly, the absolute value of (5.42) is less than a constant multiple of

$$n^{-3}\int\limits_0^{4\pi\sqrt{(n\delta)}} \theta^{\frac{5}{2}}\{1-\log(\theta n^{-\frac{1}{2}})\}\,d\theta = n^{-3}(n\delta)^{\frac{7}{4}}(A+B\log n+C\log\delta).$$

But each of the series

$$\sum d(n)n^{-\frac{5}{4}}, \qquad \sum d(n)n^{-\frac{5}{4}}\log n$$

is convergent, and so we see that (5.41) is $O(\delta^{\frac{7}{4}}\log\delta)$.

5.5. In view of 5.3 and 5.4, we get, on making δ tend to zero in the summation formula (5.23)

$$\begin{aligned}
&2\pi\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(xn)}\big)-2\pi\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(mn)}\big)\\
&= -\log\frac{x}{m}+2\pi\int\limits_0^{\infty}(2\gamma+\log t)\big\{\lambda_1\big(4\pi\sqrt{(xt)}\big)-\lambda_1\big(4\pi\sqrt{(mt)}\big)\big\}\,dt+\\
&\qquad +64\pi^6\sum_{n=1}^{\infty} d(n)\int\limits_0^{\infty} t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)\big\{x^2\lambda_3\big(4\pi\sqrt{(xt)}\big)-m^2\lambda_3\big(4\pi\sqrt{(mt)}\big)\big\}\,dt.
\end{aligned} \tag{5.51}$$

This last expression is readily evaluated by means of the integrals involving λ-functions which are dealt with in section 6. From (6.32) and (6.43), the second line of (5.51) becomes

$$-\log\frac{x}{m}-\frac{1}{4x}+\frac{1}{4m}. \tag{5.51 a}$$

From (6.22), the third line of (5.51) becomes

$$\frac{1}{x}\sum_{n<x} d(n)-\frac{1}{m}\Big\{\sum_{n\leqslant m} d(n)-\tfrac{1}{2}d(m)\Big\}. \tag{5.51 b}$$

But we have already proved in 4.5 that

$$2\pi\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(xn)}\big)=\frac{1}{x}\sum_{n<x} d(n)-\frac{1}{4x}-\gamma-\log x+\psi(2). \tag{5.52}$$

Hence (5.51) may be written in the form

$$\begin{aligned}
&-2\pi\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(mn)}\big)\\
&\qquad =\gamma+\log m-\psi(2)+\frac{1}{4m}-\frac{1}{m}\Big\{\sum_{n\leqslant m} d(n)-\tfrac{1}{2}d(m)\Big\},
\end{aligned}$$

or, writing the equation in its more usual form,

$$\sum_{n\leqslant m} d(n)-\tfrac{1}{2}d(m)-\tfrac{1}{4}-m(2\gamma-1+\log m)=2\pi m\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(mn)}\big). \tag{5.53}$$

5.6. *An alternative procedure.*

The proofs we have given in this section presuppose the convergence discussions of section 4, and our proofs of (4.11) and of (5.53) have been step-by-step extensions of the preliminary formula, namely (2.42), in which $\alpha > 2$.

We can, by omitting all reference to uniform convergence, proceed directly from (2.42) to our final formulae. For example, to prove (5.53) and also (4.11) in the case when $\alpha > 1$ and x is a positive integer, m say, we may proceed thus: modify (4.33) by writing

$$\int r_1(t)f''(t)\,dt = r_2(t)f''(t) - \int r_2(t)f'''(t)\,dt.$$

Substitute the appropriate series for $r_2(t)$ in the integral and put

$$f(t) = \Gamma(\alpha+1)\lambda_\alpha\big(4\pi\sqrt{(xt)}\big) - \Gamma(\beta+1)\lambda_\beta\big(4\pi\sqrt{(mt)}\big),$$

where $\alpha > 2$ and β is, for the moment, arbitrary.

As in section 5.5, we obtain terms in x and terms in m, and the former cancel in virtue of the truth of (4.11) for $\alpha > 2$. The remaining terms, those in m, yield the result required. The limitation $\beta > \frac{1}{2}$ appears as a condition of convergence for several integrals which arise in the course of the work. The limitation $\beta \geqslant 1$ appears as a condition of convergence for one integral only, viz.

$$\int_0^\infty J_3(\theta)J_{\beta+3}(\theta)\theta^{1-\beta}\,d\theta.$$

6. Some infinite integrals

6.1. There are many integrals, involving the functions $\lambda(x)$ and $\mu(x)$, which may be evaluated by the methods recently applied by us to integrals in the theory of Bessel functions. Of these integrals, we shall consider only such as are necessary for the work of section 5.

Let $a > 0$, $b > 0$. Consider the integral

$$\int \{K_{/\mu}(az)+K_{/\mu}(-azi)\}\{K_{/\nu}(bz)+K_{/\nu}(-bzi)\}z^5\,dz. \tag{6.11}$$

'Increasing the argument by $\frac{1}{2}\pi$'* gives

$$\int_0^\infty \{K_{/\mu}(ax)+K_{/\mu}(-axi)\}\{K_{/\nu}(bx)+K_{/\nu}(-bxi)\}x^5\,dx \tag{6.12}$$

$$= e^{3\pi i}\int_0^\infty \{K_{/\mu}(axi)+K_{/\mu}(ax)\}\{K_{/\nu}(bxi)+K_{/\nu}(bx)\}x^5\,dx,$$

* Dixon and Ferrar, *Quart. J. of Math.* (Oxford), 1 (1930), 122–45, where the implications of the phrase are explained in detail.

provided that

(i) the integrals in (6.12) are convergent,

(ii) the integral (6.11), taken over the circular arc $z = re^{i\theta}$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, tends to zero as r tends either to zero or to infinity.

From the asymptotic formula (3.42), we have, for $z = Re^{i\theta}$ and large values of R,

$$|K_{/\mu}(az)+K_{/\mu}(-azi)| \qquad (6.13)$$
$$= O(R^{-4})+O(R^{-\mu-\frac{1}{2}}e^{-aR\cos\theta})+O(R^{-\mu-\frac{1}{2}}e^{-aR\sin\theta}).$$

When $z = \delta e^{i\theta}$ and δ is small,

$$|K_{/\mu}(az)+K_{/\mu}(-azi)| = O(\log\delta).$$

Accordingly, the conditions necessary to the proof of (6.12) are satisfied if

$$\mu, \nu > \tfrac{1}{2}, \qquad \mu+\nu > 4. \qquad (6.14)$$

In this case, then, on rearranging the integrands in (6.12), we have

$$\int_0^\infty \{\lambda_\mu(ax)+iJ_{/\mu}(ax)\}\{\lambda_\nu(bx)+iJ_{/\nu}(bx)\}x^5\,dx$$
$$= -\int_0^\infty \{\lambda_\mu(ax)-iJ_{/\mu}(ax)\}\{\lambda_\nu(bx)-iJ_{/\nu}(bx)\}x^5\,dx. \qquad (6.15)$$

Equating the real parts of (6.15) gives

$$\int_0^\infty \lambda_\mu(ax)\lambda_\nu(bx)x^5\,dx = \frac{2^5}{a^\mu b^\nu}\int_0^\infty \frac{J_\mu(ax)J_\nu(bx)}{(\frac{1}{2}x)^{\mu+\nu-5}}\,dx, \qquad (6.16)$$

a well-known discontinuous integral whose value may be written down at once.

6.2. We are now in a position to evaluate the integral

$$64\pi^6y^2\int_0^\infty t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)\lambda_3\big(4\pi\sqrt{(yt)}\big)\,dt, \qquad (6.21)$$

needed in section 5. Put $\theta^2 = 16\pi^2yt$; (6.21) becomes

$$\frac{1}{32y}\int_0^\infty \lambda_2\Big(\theta\sqrt{\frac{n}{y}}\Big)\lambda_3(\theta)\theta^5\,d\theta = \frac{1}{n}\int_0^\infty J_2\Big(\theta\sqrt{\frac{n}{y}}\Big)J_3(\theta)\,d\theta,$$

whose values are known to be*

$$\frac{1}{y}, \qquad \frac{1}{2y}, \qquad 0, \qquad (6.22)$$

according as $n <, =, > y$.

* Watson, *Theory of Bessel Functions*, p. 406 (8), is the most suitable formula to use.

6.3. Again, increasing the argument by $\frac{1}{2}\pi$ gives

$$\int_0^\infty \theta\{\lambda_\alpha(\theta)+iJ_{/\alpha}(\theta)\}\,d\theta = -\int_0^\infty \theta\{\lambda_\alpha(\theta)-iJ_{/\alpha}(\theta)\}\,d\theta, \tag{6.31}$$

provided* that $\alpha > \frac{1}{2}$. Accordingly, if $\alpha > \frac{1}{2}$,

$$\int_0^\infty \lambda_\alpha\big(4\pi\sqrt{(yt)}\big)\,dt = \int_0^\infty \theta\lambda_\alpha(\theta)\,d\theta = 0. \tag{6.32}$$

6.4. Finally, if $\alpha > \frac{1}{2}$, increasing the argument by $\frac{1}{2}\pi$ gives

$$\int_0^\infty \theta\log\theta\{\lambda_\alpha(\theta)+iJ_{/\alpha}(\theta)\}\,d\theta = -\int_0^\infty \theta(\log\theta+\tfrac{1}{2}\pi i)\{\lambda_\alpha(\theta)-iJ_{/\alpha}(\theta)\}\,d\theta. \tag{6.41}$$

Hence, on equating the real parts,

$$\begin{aligned} 2\int_0^\infty \theta\log\theta\lambda_\alpha(\theta)\,d\theta &= -2^{\alpha-1}\pi\int_0^\infty \theta^{1-\alpha}J_\alpha(\theta)\,d\theta \\ &= -\pi/\Gamma(\alpha), \end{aligned} \tag{6.42}$$

on using the formula (1), p. 391 of Watson's *Bessel Functions*.

Put $\theta^2 = 16\pi^2 yt$ and use (6.32) to simplify the result of the substitution; then (6.42) becomes

$$\int_0^\infty \log t\lambda_\alpha\big(4\pi\sqrt{(yt)}\big)\,dt = -\frac{1}{8\pi y\Gamma(\alpha)}. \tag{6.43}$$

7. Voronoï's summation formula

7.1. In this section we return to the problem which gave rise to much of our previous work. We are now in a position to prove Voronoï's result in the following form.†

* Cf. (6.13). There is now a term $O(R^{-\alpha+\frac{1}{2}}e^{-R\cos\theta})$.

† Our object is to give a proof which, once the more obvious properties of the λ-functions have been investigated, shall be reasonably simple. This we can do by making $f''(x)$ bounded. It is possible to prove the theorem (or rather a slight modification of it) under the condition that $f(x)$ is of bounded variation in (a, b). Voronoï's own method can, in fact, be modified so as to prove the theorem under this condition: the arithmetical detail necessary for such a proof is considerable. Compare Voronoï, *Annales de l'École Normale*, (3) 21 (1904), 516–29, or, for the corresponding $r(n)$ problem, Landau, *Vorlesungen über Zahlentheorie*, vol. ii. 274–8.

THEOREM. *If $b>a>0$, and if in (a,b) $f(x)$ has a bounded second differential coefficient, then*

$$\sum_{[a+1]}^{[b]} d(n)f(n) = \tfrac{1}{2}d(b)f(b)-\tfrac{1}{2}d(a)f(a)+\int_a^b (\log t+2\gamma)f(t)\,dt+$$
$$+2\pi\sum_{n=1}^{\infty} d(n)\int_a^b \lambda_0\big(4\pi\surd(nt)\big)f(t)\,dt, \quad (7.11)$$

where $d(x)=0$ for non-integer values of x.

When in (4.33) we substitute

$$r_1(t) = 2\pi t^2 \sum_{n=1}^{\infty} d(n)\lambda_2\big(4\pi\surd(nt)\big), \quad (7.12)$$

we may,* since $f''(t)$ is bounded, invert the order of summation and integration. This gives us

$$\sum_{[a+1]}^{[b]} d(n)f(n) = \Big[r_0(t)f(t)-r_1(t)f'(t)\Big]_a^b+\int_a^b (2\gamma+\log t)f(t)\,dt+$$
$$+2\pi\sum_{n=1}^{\infty} d(n)\int_a^b t^2\lambda_2\big(4\pi\surd(nt)\big)f''(t)\,dt. \quad (7.13)$$

7.2. Before we integrate by parts in the last integral we must, for clearness in the detail of the work, note the formula, $k=1, 2,\ldots$,

$$\frac{d}{dt}\Big\{t^k\lambda_k\big(4\pi\surd(nt)\big)\Big\} = t^{k-1}\lambda_{k-1}\big(4\pi\surd(nt)\big). \quad (7.21)$$

It is easily proved from the series-expansion on using, at the appropriate point, the relation

$$\psi(s+1) = (1/s)+\psi(s).$$

7.3. Integrating by parts, we have

$$\int_a^b t^2\lambda_2\big(4\pi\surd(nt)\big)f''(t)\,dt$$
$$= \Big[t^2\lambda_2\big(4\pi\surd(nt)\big)f'(t)\Big]_a^b-\int_a^b t\lambda_1\big(4\pi\surd(nt)\big)f'(t)\,dt$$
$$= \Big[t^2\lambda_2\big(4\pi\surd(nt)\big)f'(t)-t\lambda_1\big(4\pi\surd(nt)\big)f(t)\Big]_a^b+\int_a^b \lambda_0\big(4\pi\surd(nt)\big)f(t)\,dt. \quad (7.31)$$

* The series for $r_1(t)$ is absolutely and uniformly convergent.

Moreover, by (5.52) and (5.53),

$$2\pi t\sum_{n=1}^{\infty} d(n)\lambda_1\big(4\pi\sqrt{(nt)}\big) = \sum_{n\leqslant t} d(n)-\tfrac{1}{2}d(t)-\tfrac{1}{4}-t\{\gamma+\log t-\psi(2)\}$$

$$= r_0(t)-\tfrac{1}{2}d(t), \qquad \text{by (4.31).} \tag{7.32}$$

We have, therefore,

$$2\pi\sum_{n=1}^{\infty} d(n)\int\limits_a^b t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)f''(t)\,dt$$

$$=\Big[r_1(t)f'(t)-\{r_0(t)-\tfrac{1}{2}d(t)\}f(t)\Big]_a^b+2\pi\sum_{n=1}^{\infty} d(n)\int\limits_a^b \lambda_0\big(4\pi\sqrt{(nt)}\big)f(t)\,dt. \tag{7.33}$$

The combination of the two results (7.13) and (7.33) proves our theorem.

7.4. *Modification of the formula when $a = 0$.*

It is obviously desirable to extend the formula (7.11) to the case when $a = 0$ and $b = \infty$. The extension from $a > 0$ to $a = 0$ and the extension from $b < \infty$ to $b = \infty$ introduce convergence discussions which are very much alike. Usually the extension to $b = \infty$ will necessitate the limiting of $f(t)$ in such a way that

$$\lim_{b\to\infty} d(b)f(b) = 0. \tag{7.41}$$

This is indeed obvious since our formula should hold whether b tend to infinity through the values 1, 2, 3,... or through the values $\frac{3}{2}, \frac{5}{2}, \ldots$. The final shape of the formula presents nothing of particular interest. On the other hand, *the extension to $a = 0$ changes the shape of the formula.* For this reason we consider only the latter extension.

We shall assume that $f''(t)$ is bounded in the interval $0 < \delta \leqslant t \leqslant b$, and that $f(t)$ and $tf'(t)$ tend to finite limits as t tends to zero.

We shall also require some limitation on $f(t)$ which will make

$$\lim_{\delta\to 0} 2\pi\sum_{n=1}^{\infty} d(n)\int\limits_0^{\delta} t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)f''(t)\,dt = 0. \tag{7.42}$$

Now, working as in 5.4, we find that

$$(nt)^{5/4}\lambda_2\big(4\pi\sqrt{(nt)}\big)$$

tends to zero with nt, and remains finite as nt goes to infinity. Accordingly, the left-hand side of (7.42) is less in modulus than

$$K\sum_{n=1}^{\infty}\frac{d(n)}{n^{5/4}}\int\limits_0^{\delta} t^{\frac{3}{4}}|f''(t)|\,dt, \tag{7.43}$$

where K is a fixed constant.

Hence, "$t^{3/4}|f''(t)|$ *integrable over* $(0,\delta)$" (7.44)

is a simple condition sufficient to ensure the validity of (7.42).

We shall now prove that, with the above conditions on $f(t)$,

$$\sum_{n=1}^{[b]} d(n)f(n) = \int_0^b (2\gamma+\log t)f(t)\,dt+\tfrac{1}{4}\lim_{t\to 0}\{f(t)-tf'(t)\}+\tfrac{1}{2}d(b)f(b)+$$
$$+2\pi\sum_{n=1}^{\infty} d(n)\int_0^b \lambda_0\big(4\pi\sqrt{(nt)}\big)f(t)\,dt. \tag{7.45}$$

7.5. *Proof of the modified formula.*

For values of $t<1$, the definitions (4.31) and (4.32) show that

$$r_0(t) = -t\{\gamma+\log t-\psi(2)\}-\tfrac{1}{4},$$
$$r_1(t) = -\tfrac{1}{2}t^2\{\gamma+\log t-\psi(3)\}-\tfrac{1}{4}t.$$

Accordingly, on using (7.42) and making a tend to zero in the formula (7.13), we get

$$\sum_{n=1}^{[b]} d(n)f(n) = r_0(b)f(b)-r_1(b)f'(b)+\tfrac{1}{4}\lim_{t\to 0}\{f(t)-tf'(t)\}+$$
$$+\int_0^b (2\gamma+\log t)f(t)\,dt+2\pi\sum_{n=1}^{\infty} d(n)\int_0^b t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)f''(t)\,dt. \tag{7.51}$$

Moreover, integration by parts gives

$$2\pi\sum_{n=1}^{\infty} d(n)\int_0^b t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)f''(t)\,dt$$
$$= r_1(b)f'(b)-2\pi\sum_{n=1}^{\infty} d(n)\lim_{t\to 0} t^2\lambda_2\big(4\pi\sqrt{(nt)}\big)f'(t)-$$
$$-2\pi\sum_{n=1}^{\infty} d(n)\int_0^b t\lambda_1\big(4\pi\sqrt{(nt)}\big)f'(t)\,dt$$
$$= r_1(b)f'(b)-\{r_0(b)-\tfrac{1}{2}d(b)\}f(b)+2\pi\sum_{n=1}^{\infty} d(n)\lim_{t\to 0} t\lambda_1\big(4\pi\sqrt{(nt)}\big)f(t)+$$
$$+2\pi\sum_{n=1}^{\infty} d(n)\int_0^b \lambda_0\big(4\pi\sqrt{(nt)}\big)f(t)\,dt. \tag{7.52}$$

Combining (7.51) and (7.52) we obtain (7.45).

NOTES ON THERMODYNAMICS

I. OSMOTIC PRESSURE AND STABILITY ✓

By E. A. MILNE

[Received 19 July 1930]

THE phenomenon of osmotic pressure is by no means an obvious one, and, granted that it exists, it is by no means apparent that the osmotic pressure of a dissolved substance must be positive rather than negative. Mere equilibrium considerations are in fact inadequate to establish its sign by the methods of pure thermodynamics. The following proof, which makes use of a theorem of Willard Gibbs on stability, shows that the osmotic pressure must be positive for stability. I have not met it in the literature, and, as it exhibits the extreme power of Gibbs's methods, it may be of general interest.

Consider a system in equilibrium, at a given temperature, consisting of a solution at pressure p' in contact across a semi-permeable membrane with the pure solvent at pressure p. If P is the osmotic pressure at pressure p', then

$$P = p' - p. \tag{1}$$

Consider next a similar system at the same temperature with a slightly stronger concentration of the dissolved substance. When the pressure in the solution is $p'+dp'$, let $p+dp$ be the pressure in the adjacent pure solvent. Then the increase in osmotic pressure dP is given by

$$dP = dp' - dp. \tag{2}$$

Let ρ_1 be the density of the pure solvent at pressure p. Let ρ_1', ρ_2' be the partial densities of the solvent and the solute in the solution at pressure p'. Let μ_1 be the potential of the pure solvent at p; μ_1', μ_2' the partial potentials of the solvent and solute in the solution at p'. Then for equilibrium we have the Gibbs condition

$$\mu_1 = \mu_1'$$

and accordingly

$$d\mu_1 = d\mu_1'. \tag{3}$$

By the theory of fundamental equations at constant temperature

$$dp = \rho_1 d\mu_1 \tag{4}$$

$$dp' = \rho_1' d\mu_1' + \rho_2' d\mu_2'. \tag{5}$$

We must now disentangle the effect of hydrostatic pressure on P from the effect of change of concentration of the solute. To do this we must measure the osmotic pressure at the same pressure p' at two different concentrations. Hence we must reduce the pressure in the second system, until the solution is at its original pressure p', i.e. we choose

$$dp' = 0,$$

and accordingly by (5)

$$\rho_1' d\mu_1' + \rho_2' d\mu_2' = 0. \tag{6}$$

We have now

$$\begin{aligned} dP &= -dp \\ &= -\rho_1 d\mu_1 \\ &= -\rho_1 d\mu_1' \\ &= \frac{\rho_1 \rho_2'}{\rho_1'} d\mu_2' \end{aligned} \tag{7}$$

on using (2), (4), (3), and (6). But by a theorem of Willard Gibbs* a necessary condition for stability is that μ_2', the partial potential of the solute, must increase as its partial density ρ_2' increases. Hence $d\mu_2' > 0$ and hence $dP > 0$. But $P = 0$ when $\rho_2' = 0$. Hence $P > 0$. This is the desired result.

II. EFFECT OF TOTAL PRESSURE ON OSMOTIC PRESSURE

Willard Gibbs's great memoir, *The Equilibrium of Heterogeneous Substances*, rarely descends from general theory to examples, and to this fact may be attributed the obscurity usually felt on reading Gibbs. The construction of simple examples may help to remove these obscurities. With this object I propose to give a short derivation, using Gibbs's methods, of the formula for the effect of total pressure on osmotic pressure.

Consider a specimen of a solution at a certain temperature and at a given pressure p'. There will be a certain pressure p at which a specimen of the pure solvent at the same temperature would be in equilibrium with this solution across a semi-permeable membrane. If P is the osmotic pressure of the solute at total pressure p', then by definition

$$P = p' - p. \tag{1}$$

Now let the total pressure on the solution be increased to $p' + dp'$, the masses of solute and solvent, and hence the mass-concentration,

* *Coll. Works*, 1928, vol. i, p. 112.

being kept constant. Let $p+dp$ be the pressure at which a specimen of the pure solvent would be in equilibrium with this solution. Then

$$dP = dp' - dp. \tag{2}$$

Now let μ_1, μ_1', μ_2' be the potentials of the pure solvent, the solvent in solution and the solute in solution, at any given pressures. Let v, v' be the volumes of given quantities of the pure solvent and solution, m_1, m_1', and m_2' the masses of pure solvent, of solvent in solution and of solute in these volumes respectively. By the definitions of the μ's as partial potentials, we have for the differential of the free-energy function F (Gibbs's ψ)

$$dF = -SdT + vdp + \sum \mu_r dm_r$$

for any phase, S being the entropy and T the temperature. Hence, if T, p, and the m's are taken as independent variables, we have in our cases

$$\frac{\partial \mu_1}{\partial p} = \frac{\partial v}{\partial m_1} \tag{3}$$

$$\frac{\partial \mu_1'}{\partial p'} = \frac{\partial v'}{\partial m_1'}. \tag{4}$$

In our case we have T, m_1', and m_2' constant, and we can adjust the quantity of the pure solvent in equilibrium across the semi-permeable membrane so that m_1 is constant. Hence

$$d\mu_1 = \frac{\partial \mu_1}{\partial p} dp = \frac{\partial v}{\partial m_1} dp \tag{5}$$

$$d\mu_1' = \frac{\partial \mu_1'}{\partial p'} dp' = \frac{\partial v'}{\partial m_1'} dp'. \tag{6}$$

But we have always $\mu_1 = \mu_1'$, and hence $d\mu_1 = d\mu_1'$. Hence

$$\frac{\partial v}{\partial m_1} dp = \frac{\partial v'}{\partial m_1'} dp'. \tag{7}$$

Hence, by (2),

$$dP = dp'\left[1 - \frac{\partial v'}{\partial m_1'} \Big/ \frac{\partial v}{\partial m_1}\right]. \tag{8}$$

But $\partial v/\partial m_1$, the increase in volume per unit increase in mass of the pure solvent is $1/\rho(p)$, where $\rho(p)$ is the density of the pure solvent at pressure p. Also, $\partial v'/\partial m_1'$ is the rate of increase in volume of the solution with respect to the mass of the solvent when the pressure p' and mass m_2' of solute are kept constant. This may be measured by the addition of one gram of the solvent to a large volume of the

solution, at its given concentration, at pressure p'. Call this $u(p')$, an observable quantity. Then

$$\left(\frac{dP}{dp'}\right)_{m_1,\, m_2} = 1 - \rho(p)u(p').$$

This is the desired formula. It should be noted that the concentration measured in quantity of solute per unit *volume* of the solution does not necessarily remain constant during the increase of pressure. It is the concentration measured in mass of solute per unit mass of solution which remains constant.

ON A QUESTION RELATED TO WARING'S PROBLEM

By R. D. CARMICHAEL

[Received 9 August 1930]

1. Multiplicative domains numerically defined. Let the totality of prime numbers be separated in any way into a finite number k of classes $C_1, C_2, \ldots, C_k$. With each class C_i associate a positive integer α_i for $i = 1, 2, \ldots, k$. Consider the set of integers which is composed of 0, 1, the α_ith power of every prime in C_i $(i = 1, 2, \ldots, k)$ and all the integers which may be obtained from these generators by multiplication (with repetition of factors permissible). This set will constitute a multiplicative domain in the sense that the product of any two numbers in the set is itself in the set. Such a domain will be denoted by the symbol D.

If μ is the least common multiple of $\alpha_1, \alpha_2, \ldots, \alpha_k$, then D contains the μth power of every positive integer. Unless $\alpha_1 = \alpha_2 = \ldots = \alpha_k$ it will also contain other integers.

Since every positive integer is representable as a sum of $n(\mu)$ μth powers of positive integers, where $n(\mu)$ depends on μ alone, it follows that for every domain D there exists an integer $N(D)$, not greater than $n(\mu)$, such that every positive integer is representable as a sum of $N(D)$ integers belonging to D. The problem of determining $N(D)$, and of other related integers suggested by the known facts concerning Waring's problem, is perhaps of little interest in the general case. But we shall present here a special case of the domains D for which the problem appears to be not devoid of interest.

Let m be an integer greater than unity. Let D_m denote the domain D which has the following generators: 0, 1, every prime factor of m, and the νth power of every prime of the form $mx + \alpha$ where α ranges over the integers less than m and prime to m, and where ν $(= \nu_\alpha)$ is the exponent to which α belongs modulo m, so that $t = \nu$ is the least positive integer t such that $\alpha^t \equiv 1 \pmod{m}$. For such a domain D_m we propose the following questions (suggested by Waring's problem). What is the least integer $g(m)$ such that every positive integer is a sum of $g(m)$ integers belonging to D_m? What is the least integer $G(m)$ such that all but a finite number of positive integers are each expressible as a sum of $G(m)$ integers belonging to D_m? What is the

least integer $\Gamma(m)$ such that nearly all positive integers are each expressible as a sum of $\Gamma(m)$ integers belonging to D_m? [To say that 'nearly all' positive integers have a given property is to say that $\psi(n)/n \to 1$ as n becomes infinite where $\psi(n)$ is the number of positive integers not exceeding n and having the property in consideration.] It is obvious that $g(m) \geqslant G(m) \geqslant \Gamma(m)$.*

It is evident that

I. $\qquad g(2) = G(2) = \Gamma(2) = 1.$

The domain D_3 is generated by 0, 1, 3, 4, primes of the form $6x+1$ and the squares of primes of the form $6x-1$. Then, from classic theorems, it follows that the numbers in D_3 are precisely the numbers of the form x^2+3y^2. But every positive integer may be written in the form $x^2+3y^2+u^2+3v^2$. No number $6x-1$ is in D_3. From these considerations it follows that

II. $\qquad g(3) = G(3) = \Gamma(3) = 2.$

Similarly, it may be shown that the domain D_4 consists of the integers of the form x^2+y^2, and thence that

III. $\qquad g(4) = G(4) = \Gamma(4) = 2.$

Since D_6 contains every number in D_3, it follows readily that

IV. $\qquad g(6) = G(6) = \Gamma(6) = 2.$

In later sections we shall prove also the following theorems:

V. $\qquad g(8) = G(8) = \Gamma(8) = 3.$

VI. $\qquad g(12) = 3,\ G(12) = 2 \text{ or } 3,\ \Gamma(12) = 2 \text{ or } 3.$

VII. $\qquad g(24) = G(24) = \Gamma(24) = 3.$

These include all the cases in which I have succeeded in finding the values of $g(m)$, $G(m)$, and $\Gamma(m)$. These cases (except for the trivial case $m = 2$) are characterized by the fact that they involve just those values of m which are such that 2 is the greatest exponent of any integer modulo m, whence it follows that in these cases D_m contains every square. It is this fact which renders the treatment of these cases easy. I am at present unable to determine whether 2 or 3 is the correct value of either $G(12)$ or $\Gamma(12)$.

The question arises whether an algebraic form exists which, for a given m, represents just the numbers D_m and no others. We have seen that this is so for $m = 3$ and $m = 4$. The numbers D_6 are the

* It may be remarked that Hardy and Littlewood use G_1 for what is here written Γ, their Γ having an entirely different signification.

numbers of the two forms x^2+3y^2 and $2(x^2+3y^2)$. In $f_8(x,y,z,u)$, as defined in equation (2.2), we have a single form which represents just the numbers D_8. The numbers D_{12} are the numbers $\alpha f_{12}(x,y,z,u)$, where α has the values 1, 2, 3, 6, and f_{12} denotes the form defined by equations (3.1) and (3.2). The corresponding result for D_{24} was not sought. Such results for D_5 and D_{10} are given in § 5.

Owing to the difficulty of the problem here characterized, I have thought it well to include some empirical results, as follows: in § 3, concerning D_{12}; in § 5, concerning D_5, D_{10}, and D_{16}; and in § 6, concerning several other domains D_m. Particular attention should be called to the empirical evidence for the (unproved) proposition that every positive integer is a sum of two numbers belonging to D_{10}.

Finally (§ 7) a formula is given for certain cyclotomic forms associated with the domains D_m.

2. The domain D_8. The generators of D_8 are 0, 1, 2, primes of the form $8x+1$ and the squares of all other odd primes. Then D_8 contains every square. Moreover, it follows from classic theorems that every number in D_8 is simultaneously of the two forms a^2+2b^2 and $\alpha^2-2\beta^2$. It is well known that every odd positive integer is of the form $x^2+y^2+2z^2$. Since D_8 is a multiplicative domain containing 2 and every square, it follows first that every odd positive integer, and thence that every positive integer, is a sum of three numbers in D_8. Hence $g(8)\leqslant 3$. But $\Gamma(8)\geqslant 3$; for no number $8x+7$ is a sum of two numbers D_8, since the numbers of D_8 are of the forms $8x+\alpha$, $\alpha=0$, 1, 2, 4. Therefore $g(8)=G(8)=\Gamma(8)=3$.

Implicit here is also the fact that every positive integer may be written as a sum $\sigma_1+\sigma_2+\sigma_3$, where each σ_i is simultaneously of the two forms a^2+2b^2 and $\alpha^2-2\beta^2$.

We shall construct a form of the fourth degree in four variables which represents just the integers in D_8 and no others. Let λ be a primitive eighth root of unity and form the function

$$f_8(x,y,z,u)=\prod_{j=1}^{4}(x+\lambda^{2j-1}y+\lambda^{2(2j-1)}z+\lambda^{3(2j-1)}u). \tag{2.1}$$

Writing the second member of (2.1) in each of the three possible ways as a product of two products of pairs of factors, we find that

$$\begin{aligned} f_8(x,y,z,u) &= (x^2-z^2+2yu)^2+(u^2-y^2+2xz)^2 \\ &= (x^2+z^2-y^2-u^2)^2+2(xy-yz+zu+ux)^2 \\ &= (x^2+y^2+z^2+u^2)^2-2(xy+yz+zu-ux)^2. \end{aligned} \tag{2.2}$$

Moreover, this form has the multiplicative property

$$f_8(x,y,z,u)f_8(a,b,c,d) = f_8(ax-bu-cz-dy, ay+bx-cu-dz, az+by+cx-du, au+bz+cy+dx). \quad (2.3)$$

We note also the special cases

$$f_8(x,0,z,0) = (x^2+z^2)^2, \qquad f_8(x,y,-x,0) = (y^2+2x^2)^2,$$
$$f_8(x,y,x,0) = (y^2-2x^2)^2. \quad (2.4)$$

Now $f_8(1,1,0,0) = 2$, so that f_8 represents 2. It represents the square of every odd prime, as one sees from (2.4), since every prime $4x+1$ is a sum of two squares, every prime $8x+3$ is of the form y^2+2x^2, and every prime $8x+7$ is of the form y^2-2x^2. Moreover, we shall now show that f_8, as was known to Jacobi (*Werke*, vi, pp. 275–80) represents every prime of the form $8x+1$. For if p is such a prime, then integers a, α, b, β exist, a and α both odd and b and β both even, such that

$$p = \alpha^2-2\beta^2 = a^2+2b^2.$$

Then we have $(\alpha-a)(\alpha+a) = (\beta-b)^2+(\beta+b)^2$, whence it follows that integers x, y, z, u exist such that

$$\alpha-a = 2(x^2+z^2), \qquad \alpha+a = 2(y^2+u^2),$$
$$\beta-b = 2(xy+zu), \qquad \beta+b = 2(yz-xu).$$

[In the detailed proof of this use may be made of the corollary on p. 42 of my *Diophantine Analysis*.] Then $p = \alpha^2-2\beta^2 = f_8(x,y,z,u)$. It is easy to show that every odd number $f_8(x,y,z,u)$ is of the form $8t+1$. From the several results in this paragraph and from the multiplicative property of f_8 and from its several forms given in (2.2), we have the following theorem:

VIII. *In order that a positive integer n shall be representable by the form $f_8(x,y,z,u)$, it is necessary and sufficient that n shall be of the form M^2N, where M and N are positive integers and N contains no odd prime factor which is not of the form $8t+1$.*

From this it follows that $f_8(x,y,z,u)$ represents just the integers in D_8 and no others. From Theorem V it follows that *every positive integer is a sum of three numbers of the form $f_8(x,y,z,u)$.*

From Theorem VIII and equation (2.2) we draw the conclusion that a positive integer whose square is not of the form $\alpha^2-2\beta^2$ $(\beta \neq 0)$ can be written as a sum $x^2+y^2+z^2+u^2$ of four squares such that $xy+yz+zu-ux = 0$. Similarly, it may be shown that an integer whose square is not of the form a^2+2b^2 $(b \neq 0)$ can be represented

in the form $x^2+z^2-y^2-u^2$, where x, y, z, u are subject to the condition $xy-yz+zu+ux=0$.

3. The domain D_{12}. The generators of D_{12} are 0, 1, 2, 3, the primes of the form $12x+1$, and the squares of the primes of the forms $12x+\alpha$ ($\alpha=5, 7, 11$). Hence D_{12} contains every square. Since every odd positive integer is of the form $x^2+y^2+2z^2$, and since D_{12} contains 2 and every square, it follows readily that every positive integer is a sum of three numbers in D_{12}. Neither of the numbers 23 and 95 is a sum of two numbers in D_{12}. Hence $g(12)=3$. Now a number D_{12} is not congruent to 5 or 7 or 10 or 11 (mod 12); hence $G(12)\geqslant\Gamma(12)\geqslant 2$. These results establish Theorem VI.

Two of my students (Mr. J. F. Locke and Miss Frances Wolever) verified that every positive integer up to 600, with the exception of 23 and 95, is a sum of two numbers in D_{12}. This would suggest the conjecture that $\Gamma(12)=2$, if not indeed also that $G(12)=2$; but I have not been able to decide between 2 and 3 for the value of either $G(12)$ or $\Gamma(12)$.

If ω is a primitive twelfth root of unity, so that $\omega^4-\omega^2+1=0$, and we form the function

$$f_{12}(x,y,z,u)=\prod_t (x+\omega^t y+\omega^{2t}z+\omega^{3t}u), \tag{3.1}$$

where t runs over the numbers 1, 5, 7, 11, we may readily show that

$$\begin{aligned} f_{12}(x,y,z,u) &= (x^2-y^2+z^2-u^2+xz-yu)^2+(xy-yz+zu+2xu)^2 \\ &= (x^2+y^2+z^2+u^2+xz+yu)^2-3(xy+yz+zu)^2 \\ &= \tfrac{1}{4}\{(S-2A+B)^2+3(S-B)^2\} \\ &= \tfrac{1}{4}\{(S-2B+A)^2+3(S-A)^2\}, \end{aligned} \tag{3.2}$$

where $\quad S=x^2+u^2, \qquad A=y^2-2xz, \qquad B=z^2-2yu.$

We note also the special cases

$$\begin{aligned} &f_{12}(x,0,0,u)=(x^2+u^2)^2, \qquad f_{12}(x,y,x,0)=(3x^2-y^2)^2, \\ &f_{12}(x,0,z,0)=(x^2+xz+z^2)^2=\tfrac{1}{4}\{(x-z)^2+3(x+z)^2\}. \end{aligned} \tag{3.3}$$

Since a number $f_{12}(x,y,z,u)$ is of the forms $\alpha^2+3\beta^2$ and $\lambda^2-3\mu^2$, where α and β are integers or the halves of integers and λ and μ are integers, it follows that $f_{12}(x,y,z,u)$ does not represent either of the primes 2 and 3. But it does represent 2^2 and 3^2, as one sees from (3.3). From the first equation in (3.3) it follows that $f_{12}(x,y,z,u)$ represents the square of every prime $4t+1$; from the second and third equations in (3.3) we see that $f_{12}(x,y,z,u)$ represents the square of every prime of the forms $12t+11$, $6t+1$. Hence $f_{12}(x,y,z,u)$ repre-

sents the square of every prime. Since $f_{12}(x,y,z,u)$ is a sum of two squares, it does not represent any prime of the form $4t+3$; since it is of the form $\rho^2-3\sigma^2$, it does not represent any prime of either of the forms $12t+5$ or $12t+7$. Hence $f_{12}(x,y,z,u)$ can represent a prime p, only if p is of the form $12t+1$, and we shall show presently that it does represent these primes. Moreover, a further consideration of the quadratic forms of $f_{12}(x,y,z,u)$ shows that any prime factor of a number $f_{12}(x,y,z,u)$, not of the form $12t+1$, enters to an even power into that number. Now the product of two numbers $f_{12}(x,y,z,u)$ is also of the same form, as one may readily prove by means of (3.1). Thus we have the following theorem:

IX. *In order that a positive integer n shall be representable by the form $f_{12}(x,y,z,u)$, it is necessary and sufficient that n shall have the form M^2N, where M and N are positive integers and N is unity or a product of primes of the form* $12t+1$.

In order to complete the proof of the theorem it remains to be shown that every prime of the form $12t+1$ is also of the form $f_{12}(x,y,z,u)$, a fact known to Jacobi (*Werke*, vi, pp. 275–80). Such a prime is representable by each of the forms

$$a^2+b^2, \qquad \alpha^2+3\beta^2, \qquad \lambda^2-3\mu^2,$$

the representation being unique in the first two cases. In these expressions β is even and α is odd, while we may (and do) take μ even and λ odd, for if μ is odd, then λ is even, and we have $\lambda^2-3\mu^2=(2\lambda+3\mu)^2-3(\lambda+2\mu)^2$, while $\lambda+2\mu$ is even. Writing $\beta=2\gamma$, $\mu=2\rho$, we have from the equation $\alpha^2+3\beta^2=\lambda^2-3\mu^2$ the relation

$$(\lambda-\alpha)(\lambda+\alpha)=12(\gamma^2+\rho^2).$$

By properly choosing the signs of λ and α, we have $\lambda-\alpha$ divisible by 3. Hence we may write

$$\lambda+\alpha=2(\gamma_1^2+\rho_1^2), \qquad \lambda-\alpha=6(\gamma_2^2+\rho_2^2),$$

where $\gamma_1\gamma_2-\rho_1\rho_2=\gamma, \qquad \gamma_1\rho_2+\gamma_2\rho_1=\rho.$

Putting $x=\gamma_1-\gamma_2$, $y=2\rho_2$, $z=2\gamma_2$, $u=\rho_1-\rho_2$, we have

$$f_{12}(x,y,z,u)=\lambda^2-3\mu^2,$$

thus proving that every prime $12t+1$ is representable by $f_{12}(x,y,z,u)$.

From Theorem IX it follows that the integers D_{12} are just the integers $\alpha f_{12}(x,y,z,u)$, where α has the values $\alpha=1, 2, 3, 6$.

4. The domain D_{24}. The generators of D_{24} are 0, 1, 2, 3, the primes of the form $24x+1$, and the squares of the primes $24x+\alpha$, $\alpha=5, 7, 11, 13, 17, 19, 23$. A number of D_{24} is of one of the forms

$24x+\beta$ ($\beta = 1, 2, 3, 4, 6, 8, 9, 12, 16, 18$). A sum of two such numbers cannot be of the form $24x+23$. Hence $\Gamma(24) \geqslant 3$. Now D_{24} contains 2 and every square; since every positive odd integer is of the form $x^2+y^2+2z^2$, it follows readily that $g(24) = 3$. Therefore we have Theorem VII.

5. The domains D_m containing every fourth power. The domain D_2 contains all the positive integers. The remaining domains D_m which contain every square are those for which $m = 3, 4, 6, 8, 12, 24$; and these have already been treated. The remaining domains D_m which contain every fourth power are those for which

$$m = 5, 10, 15, 16, 20, 30, 40, 48, 60, 80, 120, 240.$$

For none of these domains do I know the values of the functions $g(m)$, $G(m)$, and $\Gamma(m)$.

The domain D_5 is generated by 0, 1, 5, the primes $5x+1$, the squares of the primes $5x+4$, and the fourth powers of the primes $5x+2$ and $5x+3$. A number of the form $5t+4$ cannot be a sum of fewer than four numbers of D_5, since the numbers of D_5 are of the forms $5x$ and $5x+1$; hence $\Gamma(5) \geqslant 4$. Similarly it may be shown that $\Gamma(p) \geqslant p-1$ when p is any prime number. There is some empirical evidence for believing that $g(5) = 5$, this verification having been made by two of my students, Miss Laura Kinderman (for all numbers up to 500) and Mr. C. A. Jacokes (for all numbers up to 880 and for numbers of certain linear forms up to 2,500).

The domain D_{10} is generated by 2 and the domain D_5. It is evident that $\Gamma(10) \geqslant 2$. Every number up to 600 is a sum of two numbers belonging to D_{10}, as has been verified by Misses Maurine Johnson and Alice Ream. This remarkable fact raises the question whether it is true that $g(10) = G(10) = \Gamma(10) = 2$.

That every number up to 600 is a sum of four numbers belonging to D_{16} has been verified by Mr. E. R. Ott and Miss Lura McKinley.

The remaining domains belonging to this section have not been investigated either empirically or otherwise.

Let ω be a primitive fifth root of unity, and write

$$f_5(x,y,z,u) = \prod_{i=1}^{4} (x+\omega^i y+\omega^{2i} z+\omega^{3i} u). \tag{5.1}$$

Then we have $$f_5(x,y,z,u) = A^2+AB-B^2, \tag{5.2}$$

where

$$A = x^2+y^2+z^2+u^2-xy-yz-zu, \qquad B = xy+yz+zu-xz-yu-ux.$$

We have the special cases

$$f_5(x,0,0,0)=x^4, \qquad f_5(x,y,x,0)=(x^2+xy-y^2)^2. \tag{5.3}$$

Now $f_5(x,y,z,u)$ represents the fourth power of every integer and hence the fourth power of every prime. It also represents 5, since $f_5(2,1,1,1)=5$. Now $x^2+xy-y^2=(x+\frac{1}{2}y)^2-5(\frac{1}{2}y)^2$. Then from (5.3) and the fact that every prime $10t\pm1$ is representable in the form a^2-5b^2, it follows that the square of every prime $10t\pm1$ is representable by $f_5(x,y,z,u)$. Now Jacobi (*Werke*, vi, pp. 275–80) has shown that $f_5(x,y,z,u)$ represents every prime $10t+1$. From these considerations and from the fact that the product of two numbers $f_5(x,y,z,u)$ is also of that form, we see that *every number of D_5 is representable by the form $f_5(x,y,z,u)$.*

From this it follows readily that *every number of D_{10} is representable in the form $\alpha f_5(x,y,z,u)$, where α takes the values* 1, 2, 4, 8.

6. Further empirical results. Through the kindness of several of my students I have been supplied not only with the empirical facts already given but also with the following.

Every positive integer up to 615 is a sum of six numbers of D_7 (Mr. J. K. Knipp and Miss Ada Klump). Every positive integer up to 600 is a sum of five numbers of D_9 (Miss Helen McCoy and Mrs. Fannie Martin). Every positive integer up to 600 is a sum of three numbers of D_{14} (Miss Josephine Hughes Chanler). It is easy to show that $\Gamma(14)\geqslant 3$. Every positive integer up to 1,000 is a sum of six numbers of D_{32} (Miss Eleanor Kingsley and Mr. C. B. Wright).

7. Formulas for certain cyclotomic forms. It is well known that the cyclic determinant $G_n(x)\equiv G_n(x_1,x_2,\dots,x_n)$, whose first row is $(x_1,x_2,\dots,x_n)$ is separable into linear factors

$$G_n(x)=\prod_{k=0}^{n-1}(x_1+\omega^k x_2+\omega^{2k}x_3+\dots+\omega^{(n-1)k}x_n) \tag{7.1}$$

where ω is a primitive nth root of unity.

Let d be any divisor of n, and write $n=d\nu$. Denote by $G_{n,d}(x)$ the function

$$G_{n,d}(x)\equiv\prod_{k=0}^{\nu-1}(x_1+\omega^{dk}x_2+\omega^{2dk}x_3+\dots+\omega^{(n-1)dk}x_n). \tag{7.2}$$

Then it is easy to show that

$$G_{n,d}(x)=G_\nu(x_1+x_{\nu+1}+\dots+x_{(d-1)\nu+1},\dots,x_\nu+x_{2\nu}+\dots+x_{d\nu}), \tag{7.3}$$

so that $G_{n,d}(x)$ may be written as a cyclic determinant of order ν. When $d = 1$ the function reduces to $G_n(x)$.

If $n > 2$, define $g_n(x)$ by the relation

$$g_n(x) \equiv g_n(x_1, x_2, \dots, x_n) = \prod_t (x_1 + \omega^t x_2 + \omega^{2t} x_3 + \dots + \omega^{(n-1)t} x_n), \qquad (7.4)$$

where the product is taken for t ranging over the $\phi(n)$ integers less than n and prime to n. Write

$$f_n(x) \equiv f_n(x_1, x_2, \dots, x_{\phi(n)}) = g_n(x_1, x_2, \dots, x_{\phi(n)}, 0, 0, \dots, 0). \qquad (7.5)$$

Then $f_n(x)$ is a general function of which we have had special cases in the foregoing pages, namely, the cases $n = 5, 8, 12$.

It is easy to see that

$$f_n(x) f_n(y) = f_n(z),$$

where $z_1, z_2, \dots, z_{\phi(n)}$ are certain bilinear functions of the x's and y's.

We shall now show that

$$g_n(x) = \frac{G_{n,1}(x) \,.\, \pi G_{n,pp'}(x) \dots}{\pi G_{n,p}(x) \,.\, \pi G_{n,pp'p''}(x) \dots} \qquad (n > 2), \qquad (7.6)$$

where the p's denote the different prime factors of n and where the products indicated by π extend over the combinations 2, 4, 6,... at a time of p, p', p'',... in the numerator and over the combinations 1, 3, 5,... at a time in the denominator. When every x_i except x_1 and x_2 is replaced by 0, this reduces to a classic formula.

Let us consider the linear functions

$$F_r = x_1 + \omega^r x_2 + \omega^{2r} x_3 + \dots + \omega^{(n-1)r} x_n \qquad (r = 0, 1, \dots, n-1).$$

Suppose first that r is not prime to n. Let s be the number of different prime factors of n each of which is a factor of r. Then F_r appears

$$1 + \frac{s(s-1)}{2!} + \frac{s(s-1)(s-2)(s-3)}{4!} + \dots \text{ and } s + \frac{s(s-1)(s-2)}{3!} + \dots$$

times as a factor in the numerator and denominator respectively of the fraction in (7.6), and hence, since $(1-1)^s = 0$, the factor F_r occurs equally often in numerator and denominator. When r is prime to n, the factor F_r occurs once in the numerator and is absent from the denominator. Therefore we have equation (7.6).

THE INTERACTION BETWEEN HOSTS AND PARASITES

By V. A. BAILEY

[Received 1 January 1931]

1. In the first half of 1927 Dr. A. J. Nicholson asked me to investigate mathematically the abundance of two species of animals which can maintain a steady state while interacting in the following manner.

Each mature member of the host-species lays E eggs and then dies. These eggs are then searched for by the members of the parasite-species, each traversing at random a gross effective area a during its lifetime. Any egg found in this area, if not previously discovered by some parasite, is attacked by the finder and a parasite-egg deposited in it. When each parasite has traversed the area a its search ceases. The host-eggs which have survived the search then develop into adult hosts, each of which lays E eggs, and those which have been attacked serve as food for the deposited parasite-eggs during the latter's development into adult parasites. The attacked host-eggs produce a new generation of parasites in number equal to themselves. The new generations repeat the above interaction, and are succeeded by an indefinite number of similar generations, the life-cycles of the two species always being exactly concurrent.

According to Dr. Nicholson this represents in a simplified form the essential factors which determine the abundance of animals in Nature. It is undesirable and unnecessary to consider here the grounds for this view, since these will appear in a forthcoming publication, but it is of importance to biological theory to examine the consequences of this view, and it is to be hoped that mathematicians will be able to assist in this, either by solving the resulting equations or at least by extracting from them information which is useful to the biologist.

The solution to Dr. Nicholson's problem will now be given, as an introduction to a more general one, the animals being supposed uniformly distributed.

We first inquire about the gross area s that has to be traversed collectively by the parasites in order that, of u_1 host-eggs initially present per unit area, the number u per unit area remain undiscovered.

The number of previously undiscovered eggs found, while ds is

traversed, is $u\,ds$. This produces the decrease $-du$ of undiscovered eggs.

Therefore $$-du = u\,ds,$$
and so $$s = \log(u_1/u), \tag{a}$$
since $u = u_1$ when $s = 0$.

In the steady state let h be the density of the hosts at the end of a generation and p the density of the parasites. At the beginning of a generation the host-density is then Eh. The gross area traversed by the p parasites is pa. Hence in (a) we may put
$$u = h, \qquad u_1 = Eh, \qquad s = pa,$$
and so $$pa = \log E. \tag{b}$$
The number of host-eggs attacked per unit area in one generation is $Eh-h$. In the steady state this reproduces the density p of parasites.

Accordingly, $$p = Eh-h.$$
Hence, by (b), $$ha = \frac{\log E}{E-1}. \tag{c}$$
When a and E are known, the steady densities p and h are given by (b) and (c) respectively.

It is interesting to record that in certain specific numerical cases Dr. Nicholson had, by purely arithmetical approximations, already arrived at results which are in good agreement with these formulae.

At his request I have investigated the same problem with the limitations to the steady state and to concurrent life-cycles removed, and with generation and interaction proceeding continuously.

This generalized problem is, then, essentially the same as that investigated by V. Volterra.*

Volterra does not consider parasites specifically, but may be regarded as having included them as a special type of preying animal. On comparing his theory with that given here, it will be found that they differ on points which may have important bearings on the reliability of his conclusions.

2. The general problem

The notation to be used is conveniently defined in two groups, as follows:

Specified Quantities.

ξ the age of an animal,

* *Rend. Reale Accad. d. Lincei,* 5 (1927), p. 3.

$\phi(\xi)$ the probability of death per unit time of an animal of age ξ, due to environmental causes. This may be termed the *coefficient* or *factor of fatality*,

$\chi(\xi)$ the probability, per unit time, that a host of age ξ give birth to a new host. This may be termed the *factor of fertility*,

$\gamma(\xi)$ the probability that an animal of age ξ be in a condition to interact with an animal of the other species. This may be termed the *factor of interaction*,

v the effective* areal rate of movement of a parasite, assumed to be constant.

Unknown Quantities.

$N_1(t)$ the density of unparasitized hosts in all stages at time t,

$N_2(t)$ the density of parasites, in all stages, at time t,

$N_{2s}(t)$ the density of parasites actually searching at time t,

$N_{1e}(t)$ the density of hosts vulnerable and exposed to attack at time t,

$A(\xi, t)$ the age-distribution function at time t,

$\beta(t)$ the birth-rate of animals at time t,

$\delta(t)$ the death-rate of animals due to environmental causes, at time t.

$p(t)$ the rate of parasitization of hosts at time t.

Subscript 1 refers to the hosts and subscript 2 to the parasites.

In order to simplify the mathematical treatment, the N_2 parasites are regarded as all searching, and account may be taken of those parasites not in a condition to search by assigning the value zero to their factors of interaction $\gamma_2(\xi)$.

It is assumed that the sex-ratio remains constant. The density N of animals and all quantities referring to animals include both sexes, e.g. v is the areal rate of movement of an average parasite, both sexes being included in making the average.

It is also assumed that when a parasite finds an unattacked host it lays in it one egg, and that only one egg can develop in a host.

We now proceed to derive the fundamental equations, bearing in mind that the reasoning throughout is referred to unit area.

* If u_1 be the areal velocity of the host and u_2 the areal velocity of the parasite, then v is $\sqrt{(u_1^2+u_2^2)}$ by analogy with Maxwell's theory of the mean free path of a particle in a gas. Any actual variation of v with ξ can be included in $\gamma_2(\xi)$. The effect of seasonal variations can be considered by regarding ϕ, χ, γ, v as functions of t as well as of ξ.

The Hosts.

By definition,

$$\int_0^\infty A_1(\xi,t)\,d\xi = 1. \tag{1}$$

The $N_1(t)A_1(\xi,t)\,d\xi$ hosts of ages ξ to $\xi+d\xi$ are the *survivors* of the $n_1 = \beta_1(t-\xi)\,d\xi$ hosts born in the interval $t-\xi$ to $t-\xi-d\xi$. Let $n(x)$ be the number of these n_1 hosts which survive to the age x, i.e. to the time $t-\xi+x$. In the interval dx they encounter parasites of ages η to $\eta+d\eta$, of density $N_2(t-\xi+x)A_2(\eta,t-\xi+x)\,d\eta$; the number of *interactions* in time dx is therefore $v\,dx\,n(x)\gamma_1(x)N_{2s}(t-\xi+x)$, and the number of *deaths* is $\phi_1(x)n(x)\,dx$.

Plainly the births at this time cannot affect the number $n(x)$, since this refers to hosts of *age* x.

Therefore

$$-dn(x) = \{v\gamma_1(x)N_{2s}(t-\xi+x)+\phi_1(x)\}n(x)\,dx,$$

and so

$$n(\xi) = n_1 \exp\int_0^\xi -\{v\gamma_1(x)N_{2s}(t-\xi+x)+\phi_1(x)\}\,dx.$$

These are the hosts which survive parasitization and death up to the time t. Therefore

$$N_1(t)A_1(\xi,t) = \beta_1(t-\xi)f_1(\xi)\exp\int_0^\xi -v\gamma_1(x)N_{2s}(t-\xi+x)\,dx \tag{2}$$

where

$$f_1(\xi) = \exp\int_0^\xi -\phi_1(x)\,dx.$$

It is easy to see that $f_1(\xi)$ represents the fraction of hosts born at the same time which survives to the age ξ.

This equation accounts for the existence of the hosts at time t in terms of the events occurring at previous times.

The birth-rate due to the hosts of ages between ξ and $\xi+d\xi$ is

$$\chi_1(\xi)N_1(t)A_1(\xi,t)\,d\xi,$$

so

$$\beta_1(t) = N_1(t)\int_0^\infty \chi_1(\xi)A_1(\xi,t)\,d\xi. \tag{3}$$

Similarly, the death-rate of all the hosts at time t is

$$\delta_1(t) = N_1(t)\int_0^\infty \phi_1(\xi)A_1(\xi,t)\,d\xi. \tag{4}$$

The rate of parasitization of hosts of ages between ξ and $\xi+d\xi$, by parasites of ages between η and $\eta+d\eta$, is

$$vN_2(t)A_2(\eta,t)\gamma_2(\eta)\,d\eta\,N_1(t)A_1(\xi,t)\gamma_1(\xi)\,d\xi.$$

Thus the total rate of parasitization at time t is

$$p = vB_1(t)B_2(t)N_1(t)N_2(t), \tag{5}$$

where

$$B(t) = \int_0^\infty \gamma(\zeta)A(\zeta,t)\,d\zeta. \tag{6}$$

The number of hosts in the range of ages ξ to $\xi+d\xi$ which are vulnerable and exposed to attack is

$$N_1(t)A_1(\xi,t)\gamma_1(\xi)\,d\xi.$$

Integrating over all ages and using (6), we obtain

$$N_{1e}(t) = B_1(t)N_1(t). \tag{7}$$

The Parasites.

By definition,

$$\int_0^\infty A_2(\eta,t)\,d\eta = 1. \tag{8}$$

The argument leading to equation (2) may also be applied to parasites, if it be remembered that they do not meet any enemies. Therefore

$$N_2(t)A_2(\eta,t) = \beta_2(t-\eta)f_2(\eta), \tag{9}$$

where

$$f_2(\eta) = \exp\int_0^\eta -\phi_2(\eta)\,d\eta.$$

We define the *age* of a parasite as the time which has elapsed since it was laid as an egg. This means that $\gamma_2(\eta)$ is zero for the early range of values of η, in which the parasite is immature.

Hence the rate of *birth* of parasites at time t is the same as the rate of parasitization of hosts at time t. So, by the definitions, $\beta_2(t) = p(t)$, and therefore the *birth-rate* of parasites is

$$\beta_2(t) = vB_1(t)B_2(t)N_1(t)N_2(t). \tag{10}$$

The *death-rate* of all the parasites at time t is

$$\delta_2(t) = N_2(t)\int_0^\infty \phi_2(\eta)A_2(\eta,t)\,d\eta. \tag{11}$$

The number of parasites in the range of ages η to $\eta+d\eta$ which are able to parasitize is $N_2(t)A_2(\eta,t)\gamma_2(\eta)\,d\eta$. Integrating over all ages and using (6), we obtain

$$N_{2s}(t) = B_2(t)N_2(t). \tag{12}$$

We are now able to set down a system of equations which suffices to determine the quantities N_{1e}, N_{2s}, β_1, β_2 when their initial values (at $t=0$) are given.

From (2) and (3) we get

$$\beta_1(t) = \int_0^\infty \chi_1(\xi)f_1(\xi)\beta_1(t-\xi)\left\{\exp\int_0^\xi -v\gamma_1(x)N_{2s}(t-\xi+x)\,dx\right\}d\xi; \quad (13)$$

from (1) and (2) we get

$$N_{1e}(t) = \int_0^\infty \gamma_1(\xi)f_1(\xi)\beta_1(t-\xi)\left\{\exp\int_0^\xi -v\gamma_1(x)N_{2s}(t-\xi+x)\,dx\right\}d\xi; \quad (14)$$

from (9) and (12) we get

$$N_{2s}(t) = \int_0^\infty \gamma_2(\eta)f_2(\eta)\beta_2(t-\eta)\,d\eta; \quad (15)$$

and from (10), (7), and (12) we get*

$$\beta_2(t) = vN_{1e}N_{2s}. \quad (16)$$

When N_{1e}, N_{2s}, β_1, β_2 have been obtained by solving this system of four equations, the other quantities N_1, A_1, N_2, A_2, δ_1, δ_2 can be determined in this order; N_1, A_1, from equation (2) with (1), N_2, A_2, from (9) with (8), and δ_1, δ_2 from (4), (11) respectively.

The above system of four equations can be reduced to a system of two by elimination of N_{2s} and N_{1e}.

3. The steady state

This is defined to be the state in which all the quantities are independent of the time t. The equations (13) to (16) then become respectively

$$1 = \int_0^\infty \chi_1(\xi)f_1(\xi)\left\{\exp -vN_{2s}\int_0^\xi \gamma_1(x)\,dx\right\}d\xi, \quad (13.1)$$

$$N_{1e} = \beta_1\int^\infty \gamma_1(\xi)f_1(\xi)\left\{\exp -vN_{2s}\int_0^\xi \gamma_1(x)\,dx\right\}d\xi, \quad (14.1)$$

$$N_{2s} = \beta_2\int_0^\infty \gamma_2(\eta)f_2(\eta)\,d\eta, \quad (15.1)$$

$$\beta_2 = vN_{1e}N_{2s}. \quad (16.1)$$

N_{2s} is obtained from (13.1), then β_2 from (15.1), N_{1e} from (16.1), and finally β_1 from (14.1).

* This evidently may also be deduced directly from the definitions.

Moreover, equation (13.1) shows that the density of searching parasites N_{2s} varies inversely as v, but otherwise depends only on the properties of the host.

From (15.1) and (16.1) we obtain

$$N_{1e} = \left\{ v \int_0^\infty \gamma_2(\eta) f_2(\eta)\, d\eta \right\}^{-1}.$$

This shows that the density N_{1e} of vulnerable hosts exposed to attack varies inversely as v, and in other ways depends only on the properties of the parasite.

4. A particular example

Consider the two species which possess the following properties:

1. the hosts all die only *at* the age L_1;

2. they are always vulnerable and exposed to attack;

3. their fertility is the same for all ages between T_1 and L_1, $L_1 - T_1$ being very small relatively to L_1, but nil for earlier ages; each produces E offspring;

4. the parasites all die only *at* the age L_2;

5. they can parasitize only between the ages T_2 and L_2.

These properties define the functions in our theory as follows:

1. $f_1(x) = 1$, when $x < L_1$,
$\quad = 0$, when $x > L_1$;

2. $\gamma_1(x) = 1$, when $x < T_1$,
$\quad = 0$, when $x > T_1$;

3. $\chi_1(x) = c$, when $T_1 < x < L_1$, with $L_1 - T_1 \to 0$,
$\quad = 0$, for other values of x;

thus $$\int_{T_1}^{L_1} \chi_1(x)\, dx = E, \text{ and } c = \frac{E}{L_1 - T_1};$$

4. $f_2(x) = 1$, when $x < L_2$,
$\quad = 0$, when $x > L_2$;

5. $\gamma_2(x) = 1$, when $T_2 < x < L_2$,
$\quad = 0$, for other values of x.

In addition let us write

$$N_{2s}(t) = Z'(t), \quad \beta_1(t) e^{vZ(t)} = X'(t).$$

The equations (13) to (15) then become

$$X'(t) = EX'(t-L_1), \tag{13.2}$$

$$N_{1e}e^{rZ} = X(t)-X(t-L_1), \tag{14.2}$$

$$Z' = \int_{T_2}^{L_2} \beta_2(t-\eta)\,d\eta, \tag{15.2}$$

$$\beta_2 = vN_{1e}Z'. \tag{16.2}$$

5. Discussion of the problem and its solution

No attempt has been made to solve the system of equations (13) to (16) for the general problem. It is to be hoped that the attention of mathematicians may be turned to them, as it appears that the nature of their solutions may be of great importance in biology. It would suffice for most purposes if no more than the general character of the solutions were obtained.

Dr. Nicholson has given me reasons for believing that this character is similar to that which he found to exist for the unsteady state in the simpler problem discussed at the beginning. It is easy to show that in this case the densities p_n, h_n at the nth generation of the parasites and hosts respectively are given by the initial densities and the following recurrence-relations:

$$\left.\begin{aligned} h_{n+1} &= Eh_n e^{-ap_n} \\ p_{n+1} &= Eh_n - h_{n+1}. \end{aligned}\right\} \tag{17}$$

All the numerical examples tried seem to show that, as n increases, the values of p_n and h_n oscillate respectively about the steady values p and h given by (b) and (c), and that these oscillations *grow in magnitude* without limit. Plausible arguments based on the relations (17) can be advanced for the generality of this result, but it has not yet been found possible to establish it rigorously. Dr. Nicholson's belief, then, amounts to this: that $N_{2s}(t)$ and $N_{1e}(t)$ are oscillating functions of t whose amplitudes grow with t.

In order to present the problem in a simple form the particular example in section 4 has been introduced, for which the corresponding system of equations is that numbered (13.2) to (16.2).

By integration of (13.2) it is easy to show that a solution for X is given by

$$X(t) = E^{t/L_1}Y(t)+A,$$

where $Y(t)$ is a positive function which is periodic in L_1, and A is an

arbitrary constant. Beyond this initial step nothing has yet been found but particular solutions.

It is especially desirable that this system of equations be investigated to see whether Dr. Nicholson's surmise is correct.

6. Volterra's equations for the problem

These are as follows:

$$\frac{dN_1}{dt} = \epsilon_1 N_1 - \gamma_1 N_1 N_2 \tag{18}$$

$$\frac{dN_2}{dt} = -\epsilon_2 N_2 + N_2 \int_{-\infty}^{t} F(t-\tau) N_1(\tau)\, d\tau, \tag{19}$$

where

$N_1 = N_1(t)$, the density of the prey (e.g. hosts),

$N_2 = N_2(t)$, the density of their enemies (e.g. parasites),

and ϵ_1, ϵ_2, γ_1 are constants.

In establishing (19) he explicitly adopts the 'general hypothesis' (*ipotesi generali*) that the age-distribution of the animals is the same at all times. It is easily shown that (18) may be true only if this hypothesis be true, and not otherwise.

In general this hypothesis is incorrect. Moreover, as may be seen in the following simple example, it is inconsistent with Volterra's own equation (19).

If $N_1 = 0$ initially, it will always be so, and therefore no new parasites are produced, since the coefficient ϵ_2 evidently refers only to their death-rate. It then follows from (19) that $N_2 = Ce^{-\epsilon_2 t}$, and so there are always some parasites existing at any instant t. As no new parasites are produced in the time t, none of the parasites existing at this instant can have an age less than t, i.e. the age-distribution varies with t.

In the theory given in section 2 no assumption whatever is made about the age-distribution, so it is instructive to compare its results with Volterra's equations.

From the definitions of β, δ, and p we deduce

$$\frac{dN_1}{dt} = \beta_1 - \delta_1 - p,$$

$$\frac{dN_2}{dt} = \beta_2 - \delta_2.$$

On using the equations (3), (4), (5), (10), (11), these become

$$\frac{dN_1}{dt} = \rho_1 N_1 - \sigma N_1 N_2, \tag{20}$$

$$\frac{dN_2}{dt} = -\rho_2 N_2 + \sigma N_1 N_2, \tag{21}$$

where

$$\rho_1 = \rho_1(t) = \int_0^\infty \{\chi_1(\xi) - \phi_1(\xi)\} A_1(\xi, t)\, d\xi,$$

$$\rho_2 = \rho_2(t) = \int_0^\infty \phi_2(\eta) A_2(\eta, t)\, d\eta,$$

$$\sigma = \sigma(t) = v B_1(t) B_2(t).$$

The equations (20) and (21) have a general resemblance to (18) and (19) respectively, but actually differ profoundly from them, since the coefficients ρ_1, ρ_2, σ are functions of the time which are known only when the whole problem has been solved.

It is evidently not possible without further examination to assert that Volterra's theory constitutes a first approximation to the solution, since his 'general hypothesis' is approximately true only for small or very slow variations of the densities N_1 and N_2, and in Nature large and rapid variations commonly exist.

ELECTRICAL NOTES

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[Received 22 August 1930]

III. THE STRUCTURE OF ELECTRONIC GROUPS IN WAVE-MECHANICS

RECENT writers have observed that Sommerfeld's idea of representing a closed group of electrons in an atom by a radial electrostatic field is more than a rough approximation in wave-mechanics. A set of electrons in Schrödinger's theory with assigned principal and azimuthal quantum numbers, and all possible equatorial quantum numbers, has a total volume density $-e \sum \psi\psi^*$ symmetrical about the nucleus.† The theorem can be generalized to a complete Main Smith sub-group,‡ or set of electrons in Dirac's theory with assigned principal and inner quantum numbers n, j and all the possible distinct equatorial quantum numbers l, in the notation used below. The object of this note is to show that the assumed configuration is one which can actually be brought about by the internal actions of the group, together with those of the nucleus and such other electrons as can be represented by a radial field. Sommerfeld§ remarked that the polar axis in Unsöld's theory can be regarded as that of an infinitely small magnetic field, which is not conclusive since it merely states the configuration which the group would take up under the field, without any internal actions.

We shall work on the assumption that each electron is in the electromagnetic field of the volume and current densities of the other electrons. It is obviously limited and unsatisfactory in many respects,|| but gives a simple view of the problem in hand. Taking a hint from the fact that the distribution of charge and current for an electron with monomial tesseral wave-functions is symmetrical about the polar axis, and the magnetic moment calculated from the current density is parallel to that axis,†† we can prove fairly generally that a sub-group, complete or incomplete, can set in a specially

† A. Unsöld, *Ann. d. Phys.* [4], 82, p. 379 (1927).

‡ D. R. Hartree, *Proc. Camb. Phil. Soc.* 25, p. 228 (1929).

§ A. Sommerfeld, *Phys. Zeitschr.* 28, p. 238 (1927).

|| Cf. E. Schrödinger, *Ann. d. Phys.* [4], 82, p. 271 (1927).

†† Hartree, l.c., p. 230. See *Nature*, 122, p. 925 (1928), for the result of similar calculations.

simple type of configuration, in which the magnetic axes of all the electrons are parallel. Consider first a single electron in a steady electromagnetic field symmetrical about the axis of z, and take cylindrical coordinates ρ, ϕ, z. Then $A_x+iA_y=iAe^{i\phi}$ and $A_z=0$, where A and the electric potential V are functions of ρ, z only. Dirac's equations then have a solution in which the four wave-functions, omitting the time factor, are

$$i\psi_1 e^{il\phi}, \qquad i\psi_2 e^{i(l+1)\phi}, \qquad \psi_3 e^{il\phi}, \qquad \psi_4 e^{i(l+1)\phi},$$

where ψ_1, ψ_2, ψ_3, ψ_4 are real functions of ρ, z. The associated volume and current densities give rise to a field of the same symmetry. Thus if one electron in, for example, a radial field has monomial wave-functions about a certain axis, a second electron added to the system can have coaxial wave-functions in a somewhat altered field, and so on.

Corresponding results follow from a more detailed specification of the interaction, the practical application of which is however limited to a small perturbation on a radial field. Consider a group of electrons of assigned principal quantum number whose undisturbed frequencies are either identical or close together. Denote Darwin's tesseral wave-functions by $\psi(u)$, where u stands for the pair of quantum numbers j, l, and write $F(u)$, $G(u)$ for the radial functions. Thus

$$\psi_1(u)=i(j-l)F(u)P_j^l e^{il\phi}, \qquad \psi_2(u)=iF(u)P_j^{l+1}e^{i(l+1)\phi},$$
$$\psi_3(u)=-(j+l)G(u)P_{j-1}^l e^{il\phi}, \qquad \psi_4(u)=G(u)P_{j-1}^{l+1}e^{i(l+1)\phi}.$$

Write $\sum_u c_m(u)\psi(u)$ for the disturbed wave-functions of the mth electron in zero approximation, summed over all possible values of u. The mth electron contributes an amount $-e\rho_m$, ej_m to the volume and current densities respectively, where

$$\begin{aligned}\rho_m=[\textstyle\sum_u c_m(u)\psi_1(u)][\sum_{u_1} c_m^*(u_1)\psi_1^*(u_1)]+&\\ +[\textstyle\sum_u c_m(u)\psi_2(u)][\sum_{u_1} c_m^*(u_1)\psi_2^*(u_1)]+&\\ +[\textstyle\sum_u c_m(u)\psi_3(u)][\sum_{u_1} c_m^*(u_1)\psi_3^*(u_1)]+&\\ +[\textstyle\sum_u c_m(u)\psi_4(u)][\sum_{u_1} c_m^*(u_1)\psi_4^*(u_1)],&\end{aligned}$$

etc., subject to the volume integral $\int \rho_m\, d\tau$ over the whole of space being unity. It is convenient to write $j_{x\pm iy}$ for $j_x\pm ij_y$, and similarly with other vectors. Put

$$\begin{aligned}\rho(u;u_1)&=\psi_1(u)\psi_1^*(u_1)+\psi_2(u)\psi_2^*(u_1)+\psi_3(u)\psi_3^*(u_1)+\psi_4(u)\psi_4^*(u_1),\\ j_{x+iy}(u;u_1)&=2[\psi_4(u)\psi_1^*(u_1)+\psi_2(u)\psi_3^*(u_1)],\\ j_{x-iy}(u;u_1)&=2[\psi_1(u)\psi_4^*(u_1)+\psi_3(u)\psi_2^*(u_1)],\\ j_z(u;u_1)&=\psi_1(u)\psi_3^*(u_1)+\psi_3(u)\psi_1^*(u_1)-\psi_2(u)\psi_4^*(u_1)-\psi_4(u)\psi_2^*(u_1),\end{aligned}$$

and let $V(u;u_1)$, $A(u;u_1)$ be the electromagnetic potentials of these latter distributions of charge and current. There are then four equations of the type $\rho_m = \sum_{u_2}\sum_{u_3} c_m(u_2)c_m^*(u_3)\rho(u_2;u_3)$, where

$$\textstyle\sum_u c_m(u)c_m^*(u)b(u) = 1, \qquad b(u) = \int \rho(u;u)\,d\tau.$$

The contribution of the mth electron to the potentials is

$$V_m = -e\textstyle\sum_{u_2}\sum_{u_3} c_m(u_2)c_m^*(u_3)V(u_2;u_3),$$

$$A_m = e\textstyle\sum_{u_2}\sum_{u_3} c_m(u_2)c_m^*(u_3)A(u_2;u_3),$$

so that if $(\delta V)_m$, $(\delta A)_m$ is the disturbance of the field in which it moves, $(\delta V)_m = \sum' V_{m'},\ldots$, a dash denoting the omission of the term $m' = m$. Let $\nu+\nu(u)$ be the undisturbed frequency of the mth electron corresponding to $\psi(u)$, and $\nu+\delta\nu_m$ a disturbed frequency of the mth electron. The theory of perturbation then gives, for the set of equations for the mth electron,

$$\textstyle\sum_u [a_m(u;u_1)+\delta(u;u_1)b(u)\{-\nu(u)+\delta\nu_m\}]c_m(u) = 0,$$

where

$$a_m(u;u_1) = \frac{e}{h}\int [\rho(u;u_1)(\delta V)_m + \tfrac{1}{2}j_{x+iy}(u;u_1)(\delta A_{x-iy})_m + \\ + \tfrac{1}{2}j_{x-iy}(u;u_1)(\delta A_{x+iy})_m + j_z(u;u_1)(\delta A_z)_m]\,d\tau$$

and $\delta(u;u_1)$ denotes unity or zero according as u and u_1 are equal or unequal. The integrand contains terms $c_{m'}(u_2)c_{m'}^*(u_3)e^{i(l-l_1+l_2-l_3)\phi}$, and only those survive for which $l-l_1+l_2-l_3 = 0$. This gives the required result that all the equations can be satisfied monomially, $c_m(u)$ vanishing except when $u = u_m$, where u_m is one of the admissible pairs of subordinate quantum numbers. We should require to have $a_m(u_m;u_1) = 0$ when $u_1 \neq u_m$, $a_m(u_m;u_m)+b(u_m)\{-\nu(u_m)+\delta\nu_m\} = 0$, where the notation is not intended to imply that u_1 is the value of u_m for the first electron, but u_1, u_2, u_3 remain current quantum numbers. The matrix element $a_m(u_m;u_1)$ contains terms $c_{m'}(u_2)c_{m'}^*(u_3)$ where $l_m-l_1+l_2-l_3 = 0$. If $l_m \neq l_1$, $l_2 \neq l_3$ and one of the factors $c_{m'}(u_2)$, $c_{m'}^*(u_3)$ must vanish. Hence we are left with one equation for each electron, giving its $\delta\nu$.